# ALGEBRA

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Titolo nota — 29/01/2016

$\boxed{A4}$ $\quad p(x) = \sum_{i=0}^{2n} a_i x^i \quad$ tale che

1) $\alpha \le a_i \le \alpha + 1 \quad \forall i = 0, 1, \dots, 2n \qquad \alpha = 2015$

2) $\exists\, \xi \in \mathbb{R} \;:\; p(\xi) = 0$

determinare il più piccolo $n$ per cui tale polinomio esiste

$\to \xi < 0 \qquad \zeta = -\xi$

$$0 = p(\xi) = p(-\zeta) = a_{2n}\zeta^{2n} + a_{2n-2}\zeta^{2n-2} + \dots + a_0 - \left(a_{2n-1}\zeta^{2n-1} + \dots\right)$$

$$a_{2n}\zeta^{2n} + \dots + a_0 = a_{2n-1}\zeta^{2n-1} + \dots + a_1\zeta \qquad (*)$$

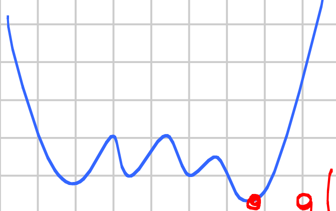
$\to$ basta studiare $q(x) = \alpha\left(x^{2n} + \dots + x^2 + 1\right) - (\alpha+1)\left(x^{2n-1} + \dots + x\right)$

a) se $q$ soddisfa 2) ok

b) se $\exists\, p$ che soddisfa 1) e 2) allora anche $q$ soddisfa 2)

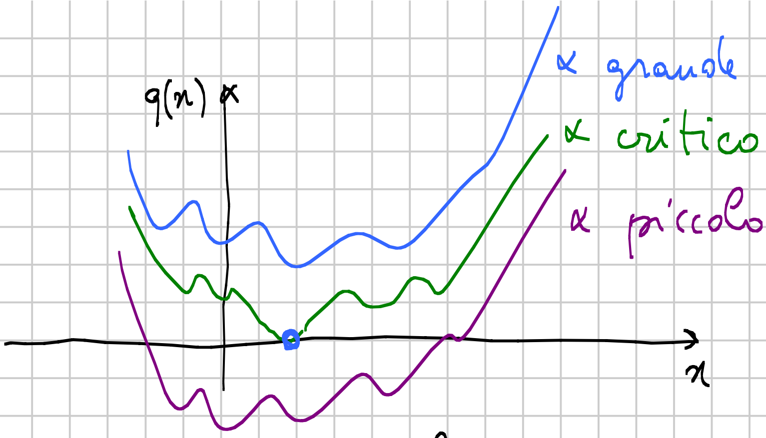
suppongo vera $(*)$ e calcolo $q(\zeta) \qquad \zeta > 0$

$$q(\zeta) = \alpha\left(1 + \zeta^2 + \dots\right) - (\alpha+1)\left(\zeta + \zeta^3 + \dots\right) \le$$
$$\le \left(a_0 + a_2\zeta^2 + \dots\right) - \left(a_1\zeta + a_3\zeta + \dots\right) = 0$$



$q(\zeta) \le 0 \implies \exists$ radice reale

$\star$ Rovescio la logica: a $n$ fissato cerco gli $\alpha$ per cui esiste $p$ (= per cui $q$ funziona)




hope: una sola radice doppia

$q$ palindromo se $\xi$ è radice, $\xi^{-1}$ pure

La radice doppia potrebbe essere 1

$$q(1) = \alpha(n+1) - (\alpha+1)\cdot n \overset{!}{=} 0 \qquad \alpha = n$$

dovrebbe essere il valore critico

$\boxed{\alpha \leqslant n \quad \text{va bene di sicuro}}$

$q_\alpha(x)$ con $\alpha = n$ $\quad q_n(x) = n(1+x^2+\dots+x^{2n}) - (n+1)(x+\dots+x^{2n-1})$

$$q_n(x) = (x-1)\left(nx^{2n-1} - x^{2n-2} + (n-1)x^{2n-3} - 2x^{2n-4} + (n-2)\dots + x - n\right)$$

$$= (x-1)^2\left(nx^{2n-2} + (n-1)x^{2n-3} + (2n-2)x^{2n-4} + (2n-4)x^{2n-5} + (3n-6)x^{2n-6} + \dots \right.$$
$$\left. \dots + (n-1)x + n\right)$$

$n \quad n-1 \quad 2n-2 \quad 2n-4 \quad 3n-6 \quad 3n-9 \quad 4n-12 \quad 4n-16$

$kn-k^2+k \quad kn-k^2$

$$q_n(x) = (x-1)^2 r(x) \qquad r(x) \geqslant 0 \text{ per } x \geqslant 0$$

$\alpha > n$

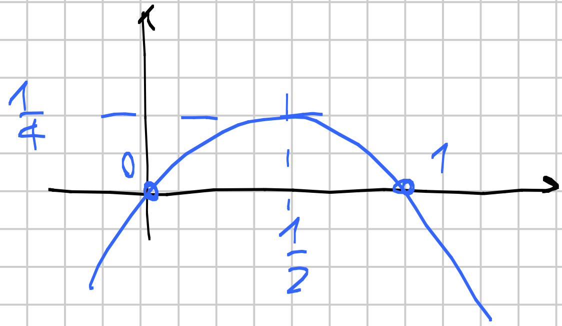
$$q_\alpha(x) = \underset{\substack{\geqslant 0 \\ =0\ x=1}}{q_n(x)} + (\alpha-n)\underbrace{(1 - x + x^2 - x^3 + \dots + x^{2n})}_{\substack{\geqslant 0 \quad >0 \quad x=1 \\ = \frac{1+x^{2n+1}}{1+x}}} > 0 \ \forall x$$

$\boxed{A5}$ Per quali $a \in \mathbb{R}$ $\exists f : (0,1] \to \mathbb{R}$ tale che $f$ non costante

$$a + f(x+y-xy) + f(x)f(y) \leq f(x) + f(y) \qquad \forall x,y \in (0,1]$$

$x = y = 1$

$$a + f(1) + f(1)^2 \leq 2f(1) \qquad a \leq f(1)(1-f(1)) \leq \frac{1}{4}$$



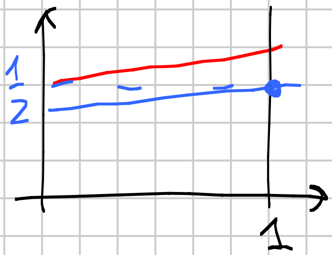

$y = 1$

$$a + \cancel{f(1)} + f(x)f(1) \leq f(x) + \cancel{f(1)}$$

$$a \leq f(x)[1-f(1)] \qquad \forall x \in (0,1]$$

$$a \leq \inf_x f(x)\,[1-f(1)]$$

Cerchiamo $f$ con massimo in $1$ e vicina alla $f \equiv \frac{1}{2}$




$$\left( f(x) = \frac{1}{2} + \varepsilon(x-1) \right)$$

$$f(x) = \frac{1}{2} + \varepsilon x$$

$$a + \frac{1}{2} + \cancel{\varepsilon x} + \cancel{\varepsilon y} - \varepsilon xy + \frac{1}{4} + \frac{1}{2}\varepsilon x + \frac{1}{2}\varepsilon y + \varepsilon^2 xy \leq 1 + \cancel{\varepsilon x} + \cancel{\varepsilon y}$$

$$a \leq \underbrace{\frac{1}{4} - \varepsilon\left(\frac{x+y}{2} - xy\right) - \varepsilon^2 xy}_{\text{cerco il minimo su } [0,1]\times[0,1]} \qquad \forall x,y$$

$(0,0)$ $\quad \frac{1}{4}$ $\qquad\qquad (0,1)$ $\quad \frac{1}{4} - \frac{\varepsilon}{2}$

$(1,1)$ $\quad \frac{1}{4} - \varepsilon^2$

Se $a = \frac{1}{4} - \frac{\varepsilon}{2}$ con $\varepsilon < \frac{1}{2}$ allora $a \leq \underbrace{\qquad\qquad}$ $\forall x,y$

Quindi $\forall a < \frac{1}{4}$ $\exists f$ che soddisfa

$$2\left(\frac{1}{4} - a\right) = \varepsilon > 0 \qquad f(x) = \frac{1}{2} + \varepsilon x \qquad \text{ecc ecc}$$

Manca $a = \frac{1}{4}$

$$a = \frac{1}{4} \Rightarrow f(1)(1 - f(1)) = \frac{1}{4} \Rightarrow f(1) = \frac{1}{2}$$

$$a \leq f(x)(1 - f(1)) \Rightarrow f(x) \geq \frac{1}{2}$$

RPA. $\exists x_0 : f(x_0) > \frac{1}{2}$

$$h(x) = 1 - f(1-x) \qquad f(x) = 1 - h(1-x)$$

$$a + \cancel{1} - h(1 - x - y + xy) + \cancel{1} - h(\cancel{1}-x) - h(\cancel{1}-y) + h(1-x)h(1-y)$$
$$\leq \cancel{2} - h(\cancel{1}-x) - h(1-\cancel{y})$$

$$a \leq h((1-x)(1-y)) - h(1-x)h(1-y) \qquad x, y \in [0,1)$$

$$x = 1-u \qquad y = 1-v$$

$$\frac{1}{4} \leq h(uv) - h(u)h(v) \qquad u, v \in (0,1]$$

$$h(u) \leq \frac{1}{2} \quad \forall u$$

RPA $\exists u_0 : h(u_0) = \frac{1}{2} - d_0 < \frac{1}{2}$

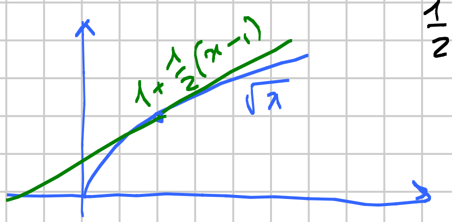
$$u_{n+1} = \sqrt{u_n}$$

$$u = v = \sqrt{u_n} \qquad \frac{1}{4} \leq h(u_n) - h(\sqrt{u_n})^2$$

$$h(u_{n+1}) = h(\sqrt{u_n}) \leq \sqrt{h(u_n) - \frac{1}{4}}$$

$$d_{n+1} = \frac{1}{2} - h(u_{n+1}) \geq \frac{1}{2} - \sqrt{\frac{1}{2} - d_n - \frac{1}{4}} = \frac{1}{2} - \sqrt{\frac{1}{4} - d_n}$$

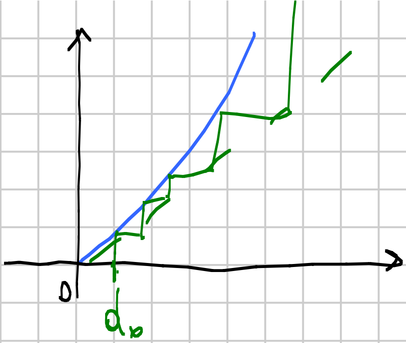
$$= \frac{\frac{1}{4} - \frac{1}{4} + d_n}{\frac{1}{2} + \sqrt{\frac{1}{4} - d_n}} = \frac{2d_n}{1 + \sqrt{1 - 4d_n}} \geq \frac{2d_n}{1 + 1 - 2d_n} = \frac{d_n}{1 - d_n}$$




$$\sqrt{1+y} \leq 1 + \frac{1}{2} y$$

$$y = \frac{x}{1-x}$$

$$= \frac{1}{1-x} - 1$$




$$d_{n+1} > d_n$$

$$\frac{d_n}{1-d_n} > \frac{d_n}{1-d_0}$$

$$d_{n+1} > d_n \lambda \qquad \lambda = \frac{1}{1-d_0} > 1$$

$$d_n > \lambda^n d_0$$

$\boxed{A6}$ $\quad a, b, c > 0 \quad , \quad a+b+c = abc$

$$abc \sum_{cyc} \frac{\sqrt{a^3+b^3}}{ab+1} \geq K \sum_{cyc} \frac{a}{a^2+1}$$

(determinare il max $K$ per cui è vera)

$a = b = c \qquad 3a = a^3 \quad a = \sqrt{3}$

hope : $K$ si ha con $a = b = c = \sqrt{3}$

Titu : $\displaystyle \sum \frac{a}{x} \geq \frac{(\sum \sqrt{a})^2}{\sum x}$

$$\sum \frac{\sqrt{a^3+b^3}}{ab+1} \geq 3 \sqrt[3]{\frac{\sqrt{(a^3+b^3)(b^3+c^3)(c^3+a^3)}}{(ab+1)(bc+1)(ca+1)}} \overset{?}{\geq} \frac{K}{abc} \sum \frac{a}{a^2+1}$$

$abc = \sum a \qquad abc(a^2+1) = a^2(a+b+c) + abc$

$$= a(a+b)(a+c)$$

$$\sum \frac{a}{a^2+1} = abc \sum \frac{1}{(a+b)(a+c)} = abc \frac{2\sum a}{(a+b)(b+c)(c+a)}$$

$$= \frac{2(abc)^2}{(a+b)(b+c)(c+a)}$$

$$abc(ab+1) = ab(a+b+c) + abc = ab(a+b+2c)$$

$$c(ab+1) = a+b+2c$$

$$abc\prod_c (ab+1) = \prod_c \big(c(ab+1)\big) = \prod_c (a+b+2c)$$

$$= \underbrace{2\sum a^3}_{3} + \underbrace{7\sum_s a^2b}_{6} + \underbrace{16abc}_{1}$$

$$3^3\sqrt{(a^3+b^3)(b^3+c^3)(c^3+a^3)}\,(a+b)^3(b+c)^3(c+a)^3$$

$$\geq K^3\, 8(abc)^3\Big[2\sum a^3 + 7\sum_s a^2b + 16abc\Big]$$

$$(a+b)(b+c)(c+a) = 2abc + \sum_s a^2b \geq 8abc$$

$$(abc)^3 = \Big(\sum a\Big)^3 = \sum a^3 + 3\sum_s a^2b + 6abc$$

$$= \frac{27}{64}\Big[\frac{64}{27}\sum a^3 + \frac{64}{9}\sum_s a^2b + \frac{128}{9}abc\Big]$$

$$\geq \frac{27}{64}\Big[2\sum a^3 + 7\sum_s a^2b + 16abc\Big]$$

bunching

diff : $\frac{10}{27}\sum a^3 + \frac{1}{9}\sum_s a^2b \geq \frac{16}{9}abc$

$$10\sum a^3 + 3\sum_s a^2b \geq 48abc$$

$$\text{LHS} \geq \text{cst}\,(abc)^7 \geq K^3(abc)^2 \geq \text{RHS}$$

$abc = \sum a \geq 3\sqrt[3]{abc}$ $\quad (abc)^3 \geq 3^3 abc$ $\quad abc \geq 3^{3/2}$