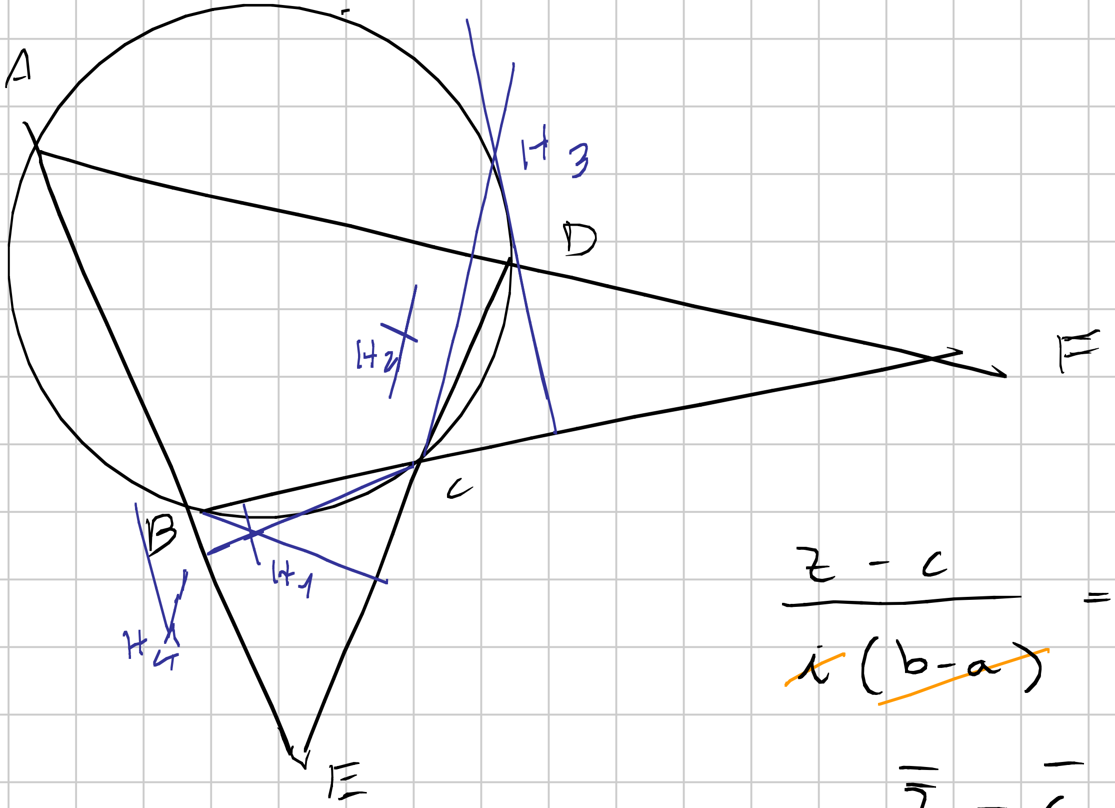
Titolo nota

WC2016 - GEOMETRIA (contosa)



$$z = c + i\lambda(b-a)$$

$$\lambda = \frac{z-c}{i(b-a)}$$

$$\frac{z-c}{\cancel{i(b-a)}} = \frac{\bar{z}-\bar{c}}{-i(\bar{b}-\bar{a})} = \frac{\bar{z}-\bar{c}}{-i\left(\frac{1}{b}-\frac{1}{a}\right)} =$$

$$= \frac{\bar{z}-\bar{c}}{\cancel{i(b-a)}} \cdot ab$$

$$z_{CH_1}:\ z - c = (\bar{z} - \bar{c})\cdot ab \quad\rightsquigarrow\quad \bar{z} = \frac{z-c}{ab} + \bar{c}$$

$$z_{BH_1}:\ z - b = (\bar{z} - \bar{b})\cdot cd$$

$$\frac{z-b}{cd} = \frac{z-c}{ab} + \bar{c} - \bar{b}$$

$$abz - ab^2 = cdz - c^2d + ab\cancel{c}\cancel{\bar{c}}d - ab\cancel{b}\cancel{\bar{b}}cd$$

$$z = \frac{ab^2 - c^2d + abd - acd}{ab - cd} = h_1$$

$$h_2 = \frac{a^2 b - c d^2 + abc - bcd}{ab - cd}$$

$$h_3 = \frac{a d^2 - c^2 b + abd - abc}{ad - bc}$$

$$h_4 = \frac{a^2 d - c b^2 + acd - bcd}{ad - bc}$$

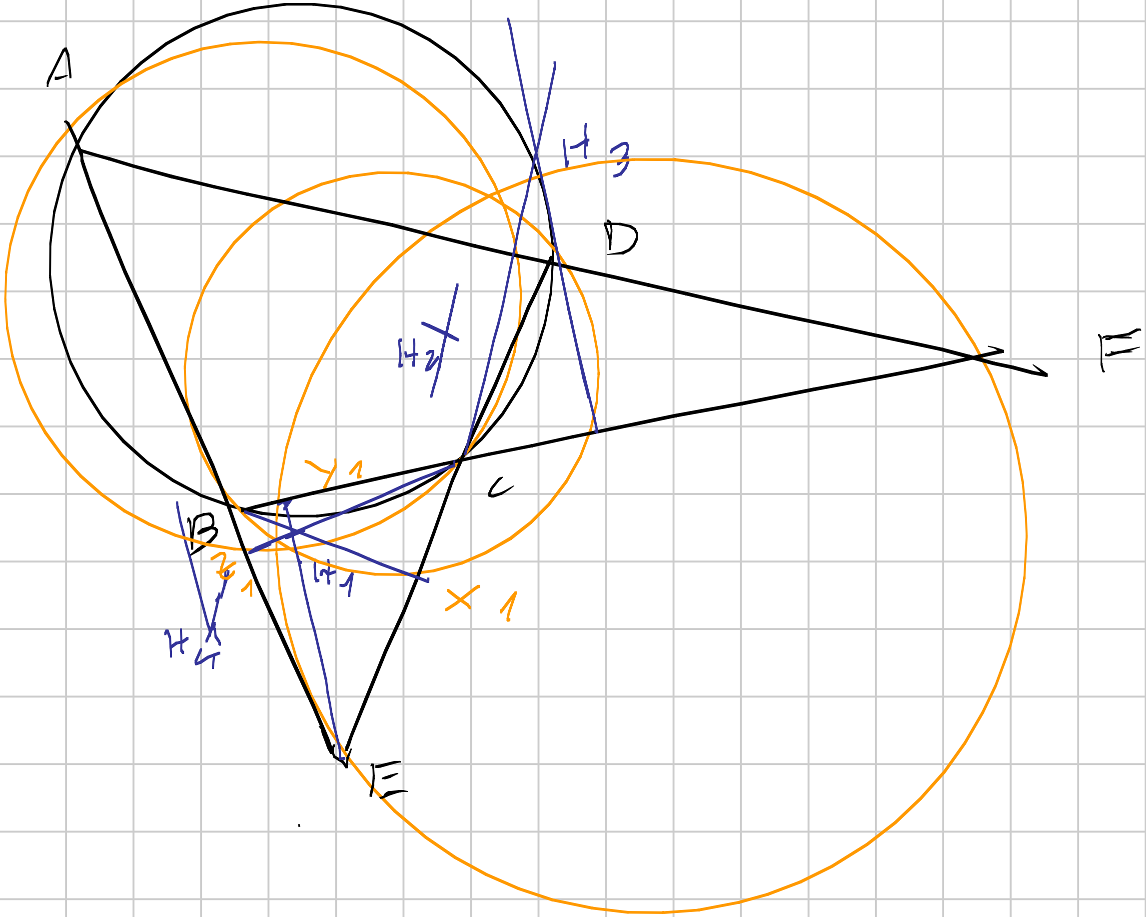
Th: $|h_3 - h_1| = |h_2 - h_4|$

$$(h_3 - h_1)(\bar{h}_3 - \bar{h}_1) = (\_\ \_\ \_\ \_\_ .)$$

HOPE $\rightsquigarrow$ $h_3 - h_1 \overset{?}{=} h_2 - h_4$

$$h_3 + h_4 \overset{?}{=} h_1 + h_2$$

$$\frac{(ab-cd)(a+b+c+d)}{(ab-cd)} \qquad \frac{(ad-bc)(a+b+c+d)}{(ad-bc)}$$



PASSO 1

$\omega_1, \omega_2, \omega_3$
di diametro
AC, BD, EF

$\Downarrow$

H_1 ha la stessa potenza rispetto a $\omega_1, \omega_2, \omega_3$

$$\text{pow}_{\omega_3}(H_1) = H_1Y_1 \cdot H_1E$$

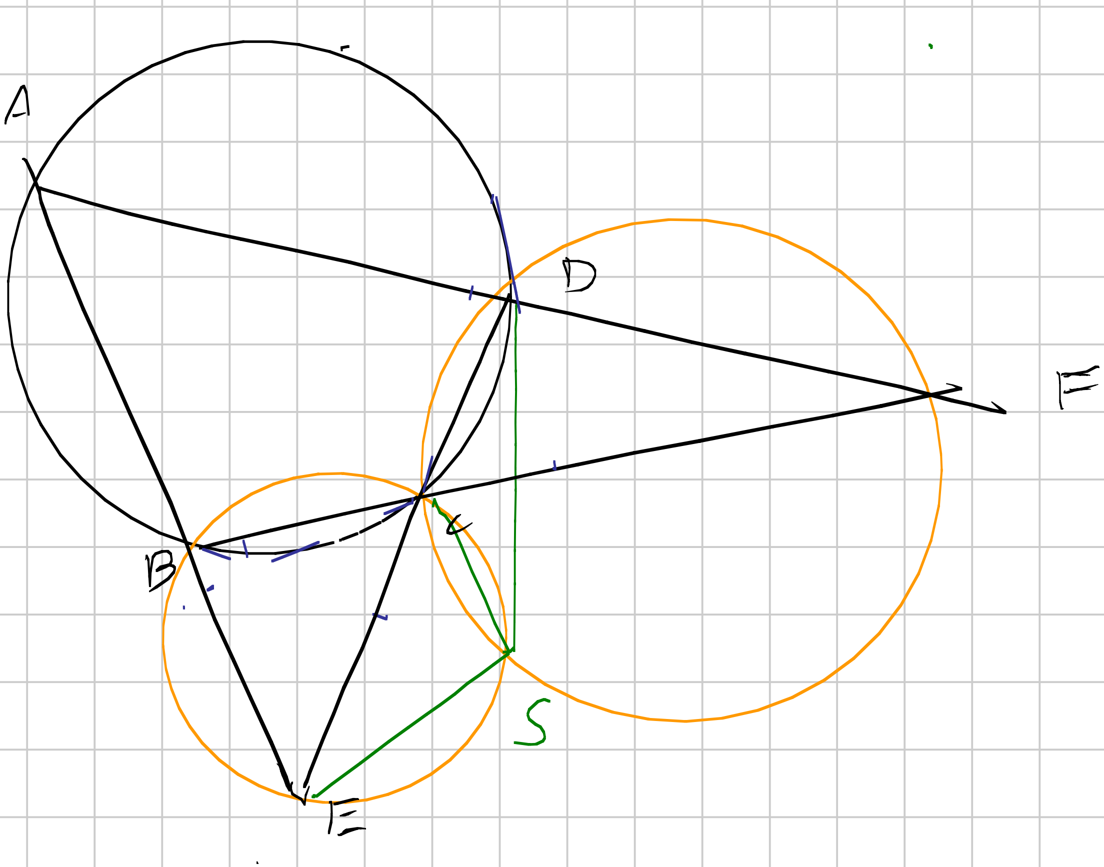
$$\text{pow}_{\omega_1}(H_1) = H_1Z_1 \cdot H_1C$$

$$\text{pow}_{\omega_2}(H_1) = H_1B \cdot H_1X_1$$

- $H_1 \; H_2 \; H_3 \; H_4 \longrightarrow$ allineati

- $\omega_1, \omega_2, \omega_3 \rightarrow$ coassiali

- centri $\longrightarrow$ allineati

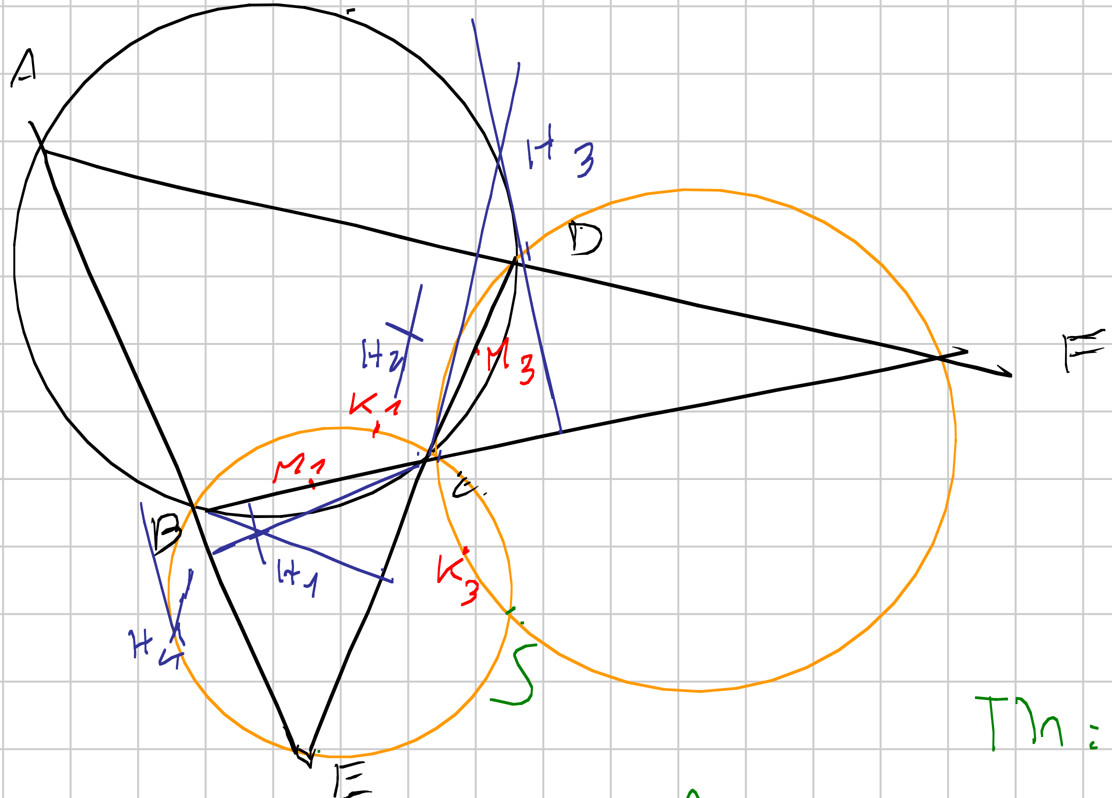


PASSO 2 (Miquel)

(BCE), (CDF),
(BFA), (ADE)

↓

concorrono in S



PASSO 3

$M_1 \to$ pt medio di BC

$K_1 \to$ simm di H_1 risp a M_1

M_3, K_3 analoghi

Th: $K_1 K_3 S$ allineati

$C\hat{S}K_1 = C\hat{B}K_1 = B\hat{C}H_1 = 90 - C\hat{B}E$

$C\hat{S}K_3 = \ldots = 90 - C\hat{D}A$

$=$

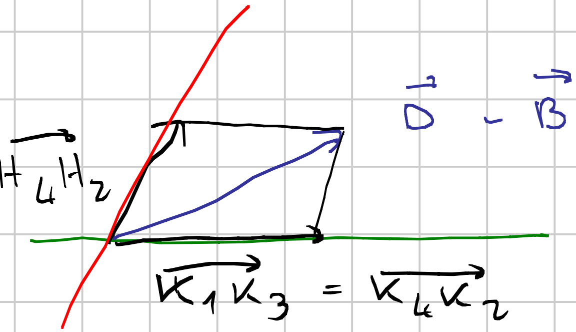
$S \quad K_1 K_2 K_3 K_4$ allineati

$$\vec{M_3} - \vec{M_1} = \frac{\vec{H_3} + \vec{K_3}}{2} - \frac{\vec{H_1} + \vec{K_1}}{2} =$$

$$= \frac{\overrightarrow{H_1 H_3}}{2} + \frac{\overrightarrow{K_1 K_3}}{2}$$

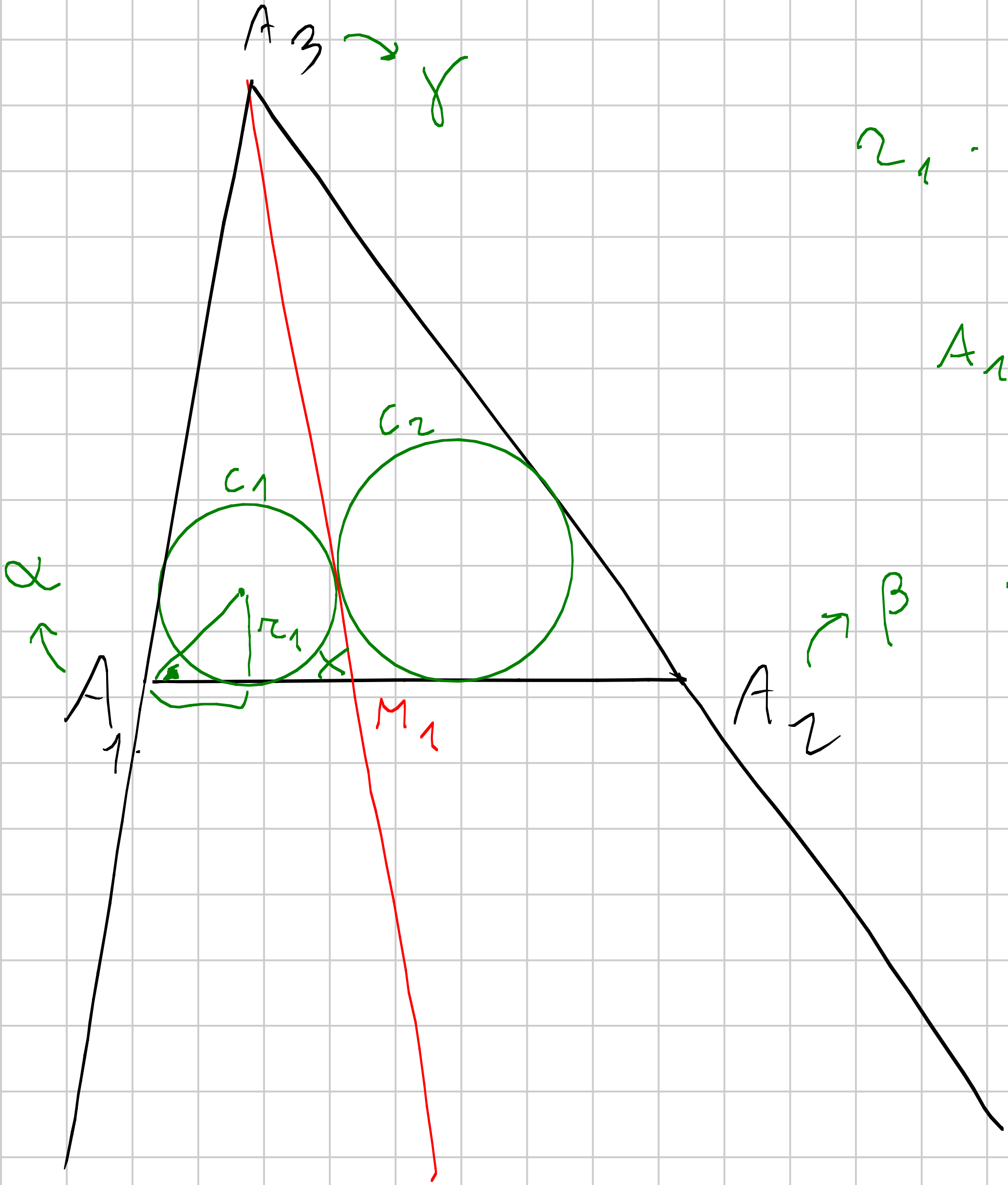
$$\overrightarrow{H_1 H_3} + \overrightarrow{K_1 K_3} = \vec{D} - \vec{B} = \overrightarrow{H_4 H_2} + \overrightarrow{K_4 K_2}$$

$\overrightarrow{H_1 H_3} = \overrightarrow{H_4 H_2}$



$\vec{D} - \vec{B}$

$\overrightarrow{K_1 K_3} = \overrightarrow{K_4 K_2}$

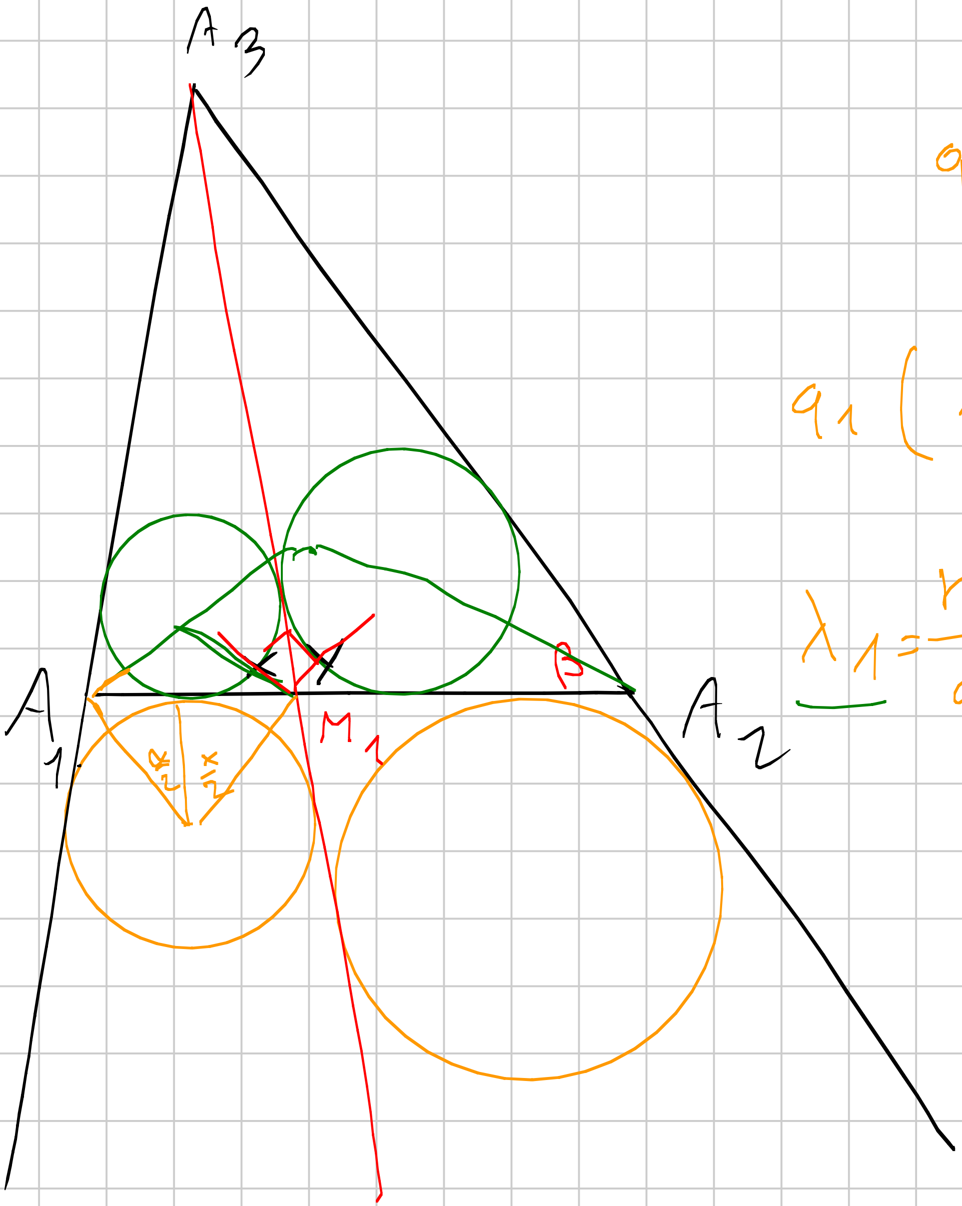


$$r_1 \cdot \cot\frac{\alpha}{2} + r_1 \cdot \cot\frac{x}{2} = A_1M_1$$

$$A_1M_1 = r_1\left(\cot\frac{\alpha}{2} + \cot\frac{x}{2}\right)$$

$$A_1M_1 = r_1\left(\frac{1}{\underbrace{\tan\frac{\alpha}{2}}_{A=}} + \frac{1}{\underbrace{\tan\frac{x}{2}}_{X}}\right) =$$

$$= r_1\left(\frac{1}{A} + \frac{1}{X}\right)$$



$$q_1 \cdot \operatorname{tg}\frac{\alpha}{2} + q_1 \cdot \operatorname{tg}\frac{x}{2} = A_1M_1$$

$$q_1(A + X) = A_1M_1 = r_1\left(\frac{1}{A} + \frac{1}{X}\right)$$

$$\lambda_1 = \frac{r_1}{q_1} = \frac{A+X}{\frac{1}{A}+\frac{1}{X}} = A \cdot X$$

$$\lambda_2 = B \cdot Y$$

$$\boxed{\operatorname{tg}\frac{x}{2} \cdot \operatorname{tg}\frac{Y}{2} = 1}$$

$$\lambda_1 \cdot \lambda_2 = A \cdot X \cdot B \cdot Y = A \cdot B = \operatorname{tg}\frac{\alpha}{2} \cdot \operatorname{tg}\frac{\beta}{2} =$$

$$= \frac{2}{t_3} = S_3$$

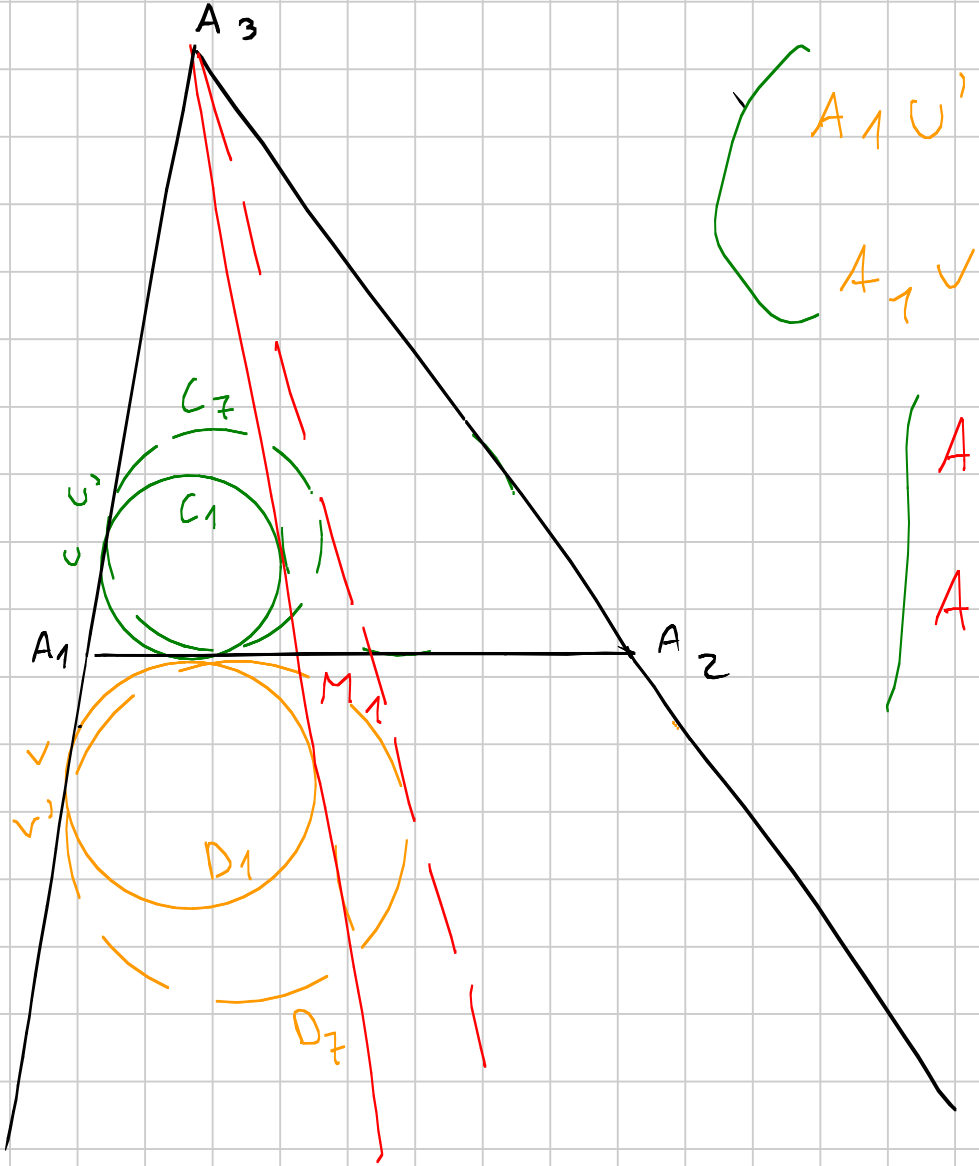
$$\lambda_2 = \frac{S_3}{\lambda_1}$$

$$\lambda_3 = \frac{S_1}{\lambda_2} = \frac{S_1}{S_3} \cdot \lambda_1$$

$$\lambda_4 = \frac{S_2}{\lambda_3} = \frac{S_2 \cdot S_3}{S_1} \cdot \frac{1}{\lambda_1}$$

$$\lambda_7 = \frac{S_2 \cdot S_3}{S_1} \cdot \frac{1}{\lambda_4}$$

$$\boxed{\lambda_7} = \frac{\cancel{S_2} \cdot \cancel{S_3}}{\cancel{S_1}} \cdot \frac{\cancel{S_1}}{\cancel{S_2} \cdot \cancel{S_3}} \cdot \boxed{\lambda_1}$$



$$\begin{cases} A_1U' > A_1U \\ A_1V' > A_1V \end{cases}$$

$$\begin{cases} A_3U' < A_3U \\ A_3V' > A_3V \end{cases}$$

$$\lambda_1 = \frac{r_1}{q_1} = \frac{A_3U}{A_3V}$$

$$\parallel \qquad\qquad \parallel$$

$$\lambda_7 = \frac{r_7}{q_7} = \frac{A_3U'}{A_3V'}$$

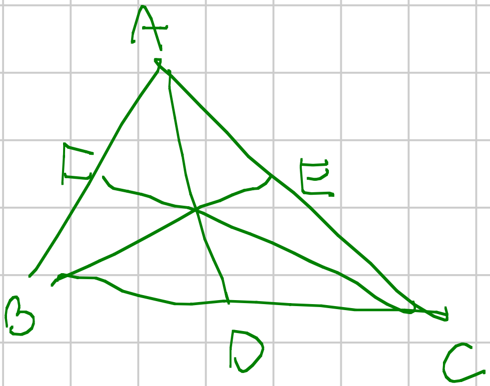
$A_3U > A_3U'$

$A_3V < A_3V'$

$$\frac{A_3U}{A_3V} > \frac{A_3U'}{A_3V'}$$

ASSURDO

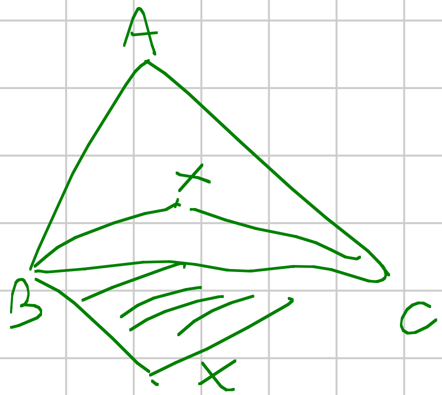
Soluzione esercizio 3



O circocentro
I incentro

<u>Th.</u> $N_{DEF} \in OI$

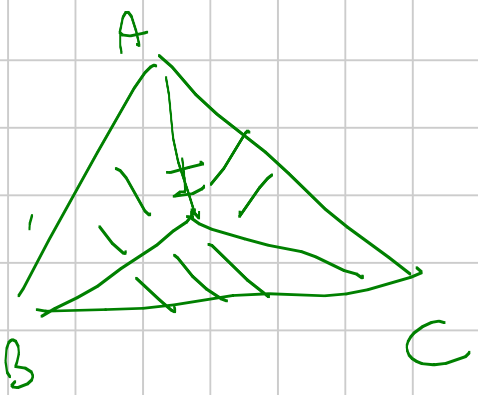
BARICENTRICHE



$$\left(\frac{[BXC]}{[BAC]} : \frac{[CXA]}{[BAC]} : \frac{[AXB]}{[BAC]} \right)$$

$$\| $$

$$(\lambda \cdot \ : \lambda \cdot \ : \lambda \cdot)$$



$$I = (a, b, c)$$

Es.:

$$O = (a^2 S_A, b^2 S_B, c^2 S_C)$$

$$S_A = \frac{b^2 + c^2 - a^2}{2} \quad \text{e cicliche}$$

$$H = (S_B S_C : S_A S_C : S_A S_B)$$

- Thread from
- Paul Yiu, Introduction to the Geometry of Triangle
- Evan Chen

Problema

O G X F

Riscriviamo il problema

ON ⊥ asse radicale fra la circoscritta e Feuerbach DEF (r_1)

OI ⊥ asse radicale fra inscritta e circoscritta (r_2)

$N G O I \iff r_1 // r_2$

Oss.

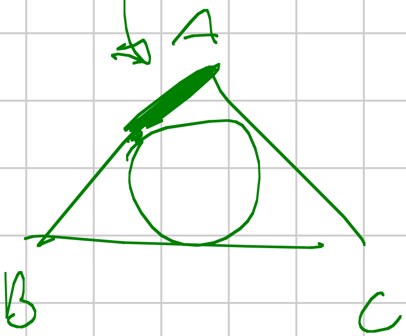
Asse radicale fra Γ (circoscritta) e γ (a caso)

$$p\,x + q\,y + r\,z = 0$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ $\qquad\qquad$ $\downarrow$

$\mathrm{Pow}_A\,\gamma \qquad \mathrm{Pow}_B\,\gamma \qquad \mathrm{Pow}_C\,\gamma$

$p - a \quad \left(= \dfrac{b+c-a}{2}\right)$



Asse radicale fra Γ e inscritta è

$$(p-a)^2 x + (p-b)^2 y + (p-c)^2 z = 0 \qquad (t)$$

$x + y + z = 0$ retta all'∞

$//$

<u>Oss 2</u> $\quad p\,x + q\,y + r\,z = 0$

$$\infty = (q - r,\ r - p,\ p - q)$$

$$\infty_E = \left((p-b)^2 - (p-c)^2 : cyc \right)$$

$$\left(a(c-b) : cyc \right)$$

Sia $\frac{\alpha}{a}x + \frac{\beta}{b}y + \frac{\gamma}{c}z = 0 \; (r)$ l'asse radicale fra Γ e Feuerbach

<u>Th</u>: $\infty_E \in (r)$

$$\sum_{cyc} \alpha(c-b) = 0$$

L'eq. della cfr in generale è

$$\underbrace{a^2yz + b^2xz + c^2xy}_{} + \underline{(x+y+z)}\left(\frac{\alpha}{a}x + \frac{\beta}{b}y + \frac{\gamma}{c}z\right) = 0$$

- $D(0, b, c)$
  $E(a, 0, c)$
  $F(a, b, 0)$

- $M_{DE}\left(a(b+c), b(a+2b+c), c(a+b)\right)$
- $M_{EF}\left(a(2a+b+c), b(a+c), c(a+b)\right)$

- $a^2bc(a+c)(a+b) + b^2ac(2a+b+c)(a+b) + c^2ab\left((2a+b+c)(a+c)\right) = 2(a+b)(a+c)\left[\alpha(2a+b+c) + \beta(a+c) + \gamma(a+b)\right]$

$$\alpha(2a+b+c)+\beta(a+c)+\gamma(a+b)=P\;\;\cancel{abc\,[\alpha(a+c)(a+b)+\beta(2a+b+c)+\gamma((a+c)(2a+\dots}$$

$$\alpha(b+c)+\beta(a+2b+c)+\gamma(a+b)=Q\;(c\leftrightarrow c \to P)$$

$$\alpha(b+c)+\beta(a+c)+\gamma(a+b+2c)=R\;(c\leftrightarrow c\to Q)$$

$$\alpha=\frac{\begin{pmatrix} P & a+c & a+b \\ Q & a+2b+c & a+b \\ R & a+c & a+b+2c \end{pmatrix}\;(*)}{\underbrace{\begin{pmatrix} 2a+b+c & a+c & a+b \\ b+c & a+2b+c & a+b \\ b+c & a+c & a+b+2c \end{pmatrix}}}=\frac{}{4(a+b)(b+c)(c+a)}$$

solo sapere che il det è simmetrico in a

$$(*)\;\;2\left[P(b+c)(a+b+c)-Qc(a+c)-Rb(a+b)\right]$$

$$\beta = \frac{\begin{pmatrix} 2a+b+c & P & a+b \\ b+c & Q & a+b \\ b+c & R & a+b+2c \end{pmatrix}}{4(a+b)(b+c)(c+a)} = \frac{2\left[Q(c+a)(a+b+c) \ldots\right]}{4(a+b)(b+c)(c+a)}$$

$$\text{Th} \iff \sum \alpha(b-c) = 0$$

$$\Updownarrow$$

$$\sum_{cyc} P(b+c)(b-c)(a+b+c) - Qc(b-c)(a+c) - Rb(b-c)(a+b) = 0$$

$$\Updownarrow$$

$$\sum_{cyc} P\left[(b+c)(b-c)(a+b+c) - c(b-c)(a+c) - b(b-c)(a+b)\right] = 0$$

$$\Updownarrow$$

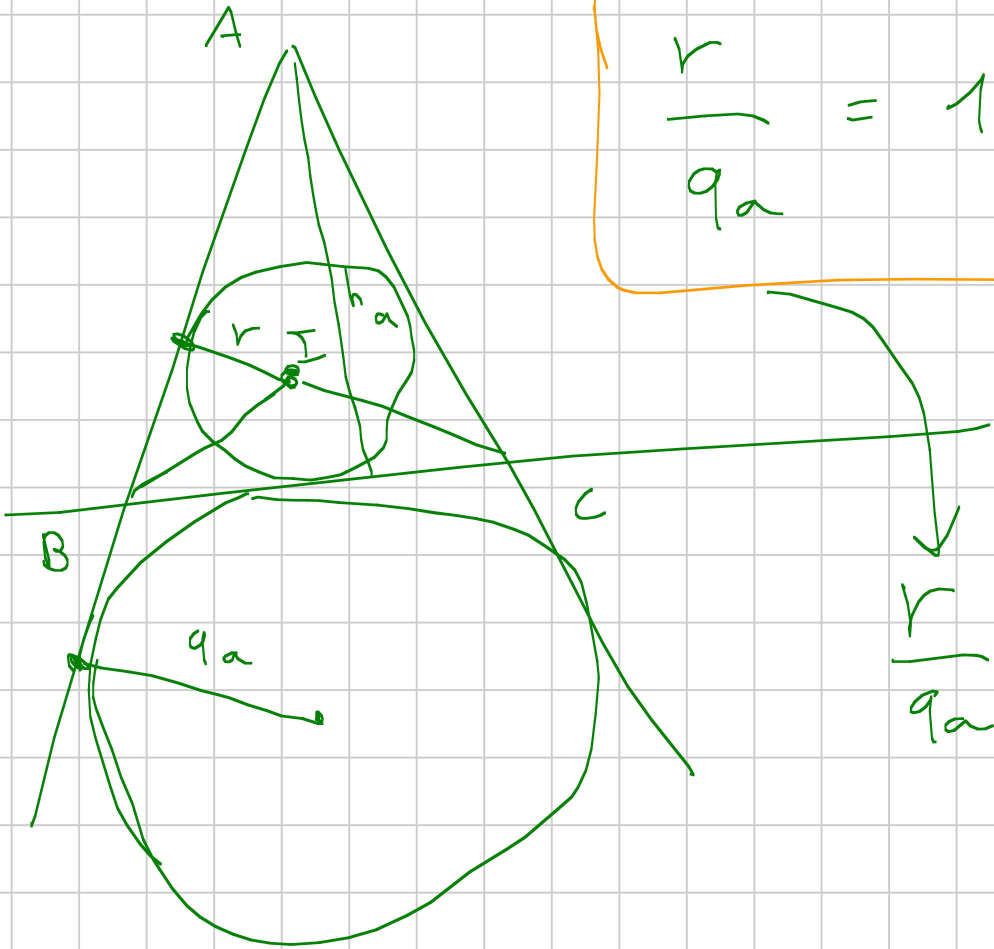
$$\sum_{cyc} P\,(b^2-c^2)(b+c) = 0$$

$$\Updownarrow$$

$$\sum_{cyc} (b+c)^2(b^2-c^2)\big[a(a+c)(a+b)+b(a+b)(2a+b+c) + c(a+c)(2a+b+c)\big] = 0$$

$$\Updownarrow$$

Vero!

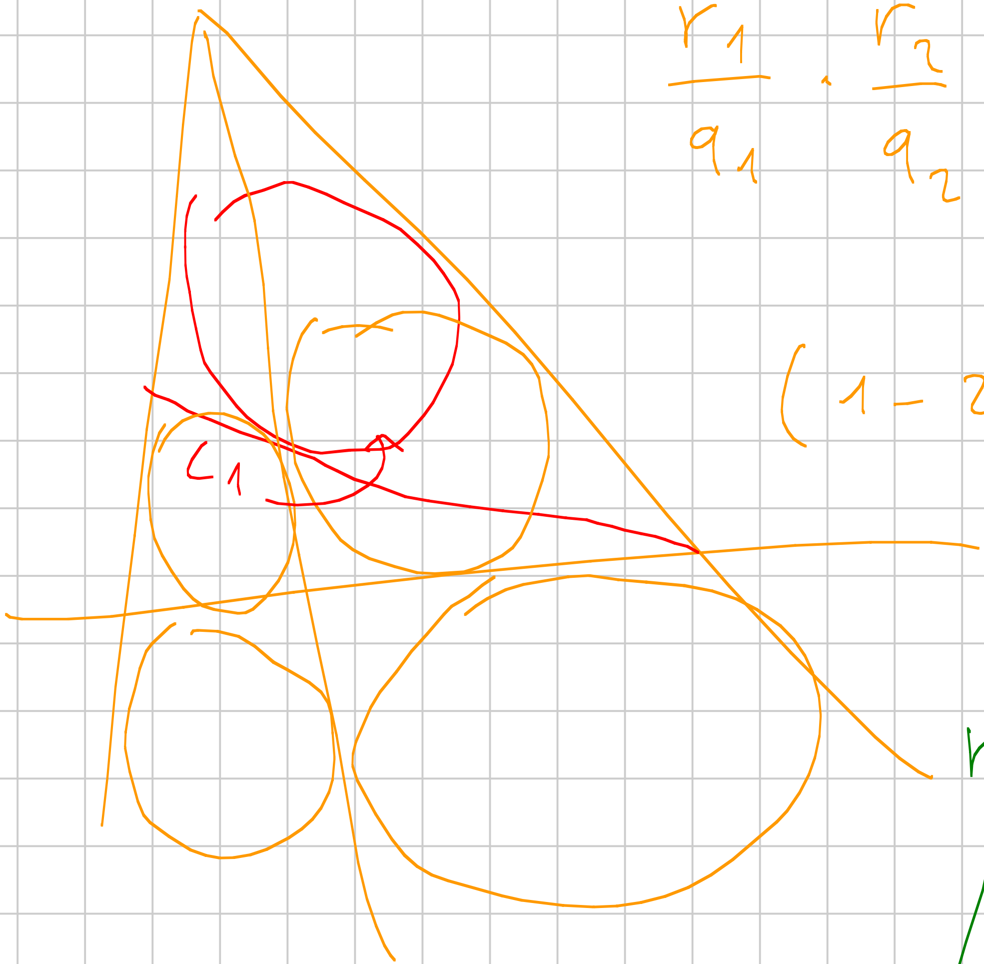


$$\boxed{\frac{r}{q_a} = 1 - 2\,\frac{r}{h_a}}$$

$$\frac{r}{q_a} = \frac{b+c-a}{b+c+a}$$

$$\frac{r}{h_a} = \frac{[IBC]}{[ABC]} = \frac{ar}{ar+br+cr} = \frac{a}{a+b+c}$$

$$\frac{b+c-a}{b+c+a} = 1 - 2\,\frac{a}{a+b+c}$$



$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q_a}$$

$$\left(1 - 2\frac{r_1}{h_a}\right)\left(1 - 2\frac{r_2}{h_a}\right) = \left(1 - 2\frac{r}{h_a}\right)$$

$$r_2 = h_a \cdot \frac{r - r_1}{h_a - 2r_1}$$

$$r_3 = h_b \cdot \frac{r - r_2}{h_b - 2r_2} =$$

$$= h_b \, \frac{r_1 (h_a - 2r)}{2r_1 (h_a - h_b) + h_a (h_b - 2r)}$$

$$r_4 = h_c \cdot \frac{r - r_3}{h_c - 2r_3}$$

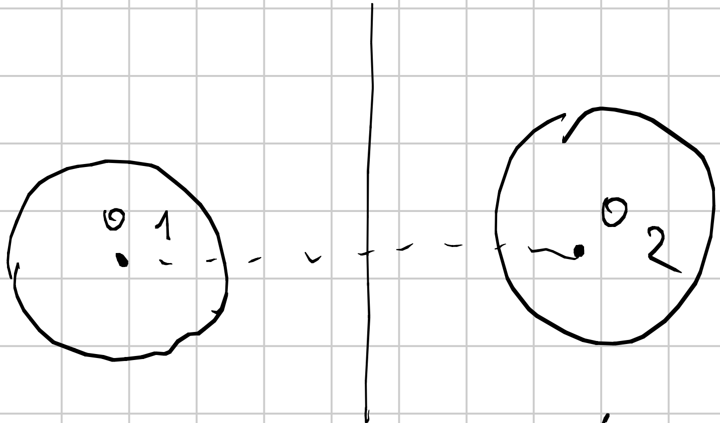
$$r_4 = \frac{h_a h_c (h_b - 2r)(r - r_1)}{h_a h_c (h_b - 2r) + 2r_1 \left(h_a h_c + h_b (2r - h_a - h_c)\right)}$$

$$r_1 \to r_0 \to r_{-1} \to r_{-2}$$

$r_{-2} \stackrel{?}{=} r_4$ ci basta che r_4 simmetrico h_a, h_c

SOL. ALTERNATIVA 3

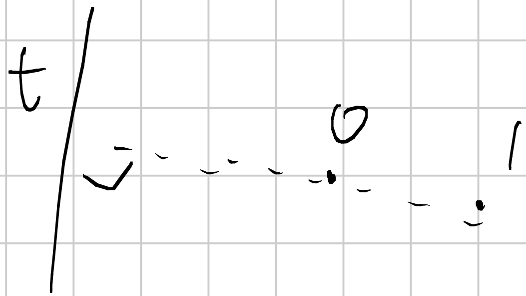
LEMMA:

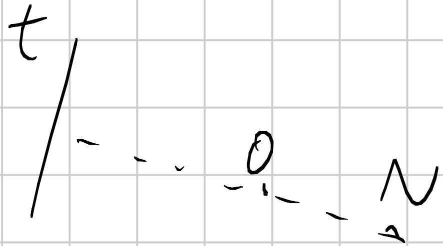


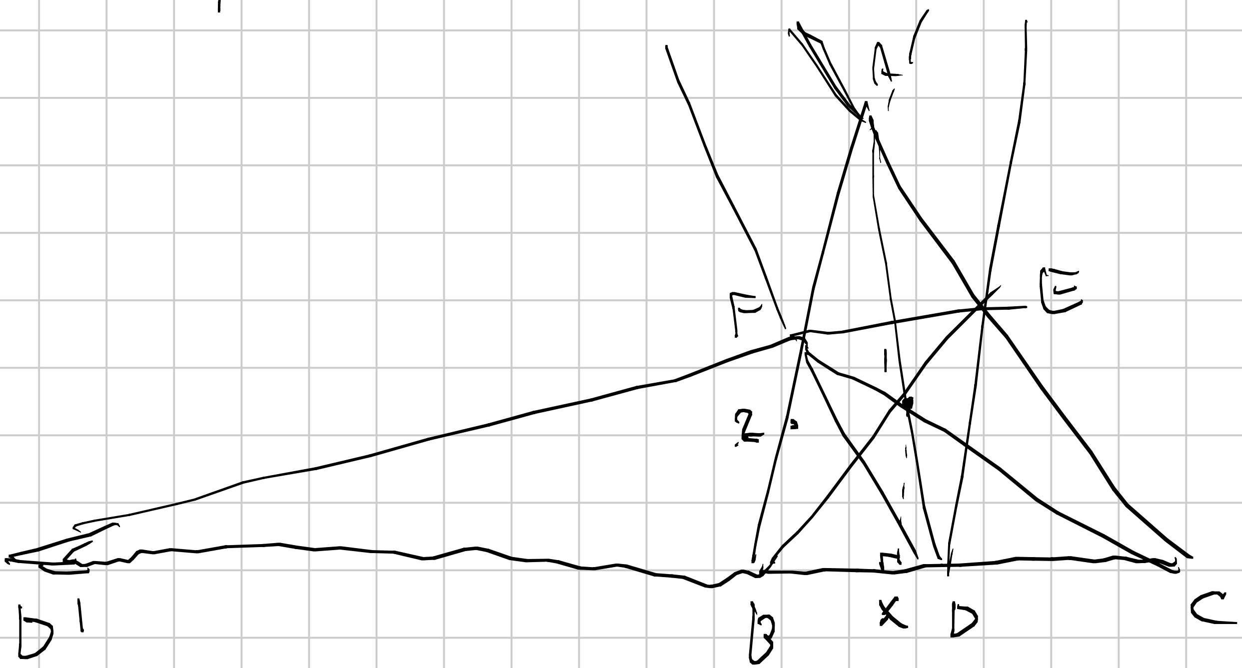
$P : pow_{(O_1)}(P) - pow_{(O_2)}(P) = cost.$

(le cost. = 0 è l'asse rad.)

in GEN. è una retta $\perp$ a $O_1 O_2$







WLOG $b > c$

$$BD' = \frac{ac}{b-c} \quad \text{perché} \quad (BCDD') = -1$$

$$BD = \frac{ac}{b+c}$$

$$pow_{(\Gamma(o))}(D') = D'B \cdot D'C = \frac{a^2bc}{(b-c)^2}$$

$$pow_{(\Gamma(1))}(D') = D'X^2 = \left(\frac{2ac}{2(b-c)} + \frac{(b-c)(a+c-b)}{2(b-c)} \right)^2$$

$$= \frac{1}{4} \left(\frac{a(b+c)}{b-c} - (b-c) \right)^2$$

$$\mathrm{pow}_{(C(I))}(D') - \mathrm{pow}_{(C(O))}(D') =$$

$$= \frac{1}{4}\left(\frac{a^2(b+c)^2}{(b-c)^2} + (b-c)^2 - 2a(b+c) - \frac{4a^2bc}{(b-c)^2} \right)$$

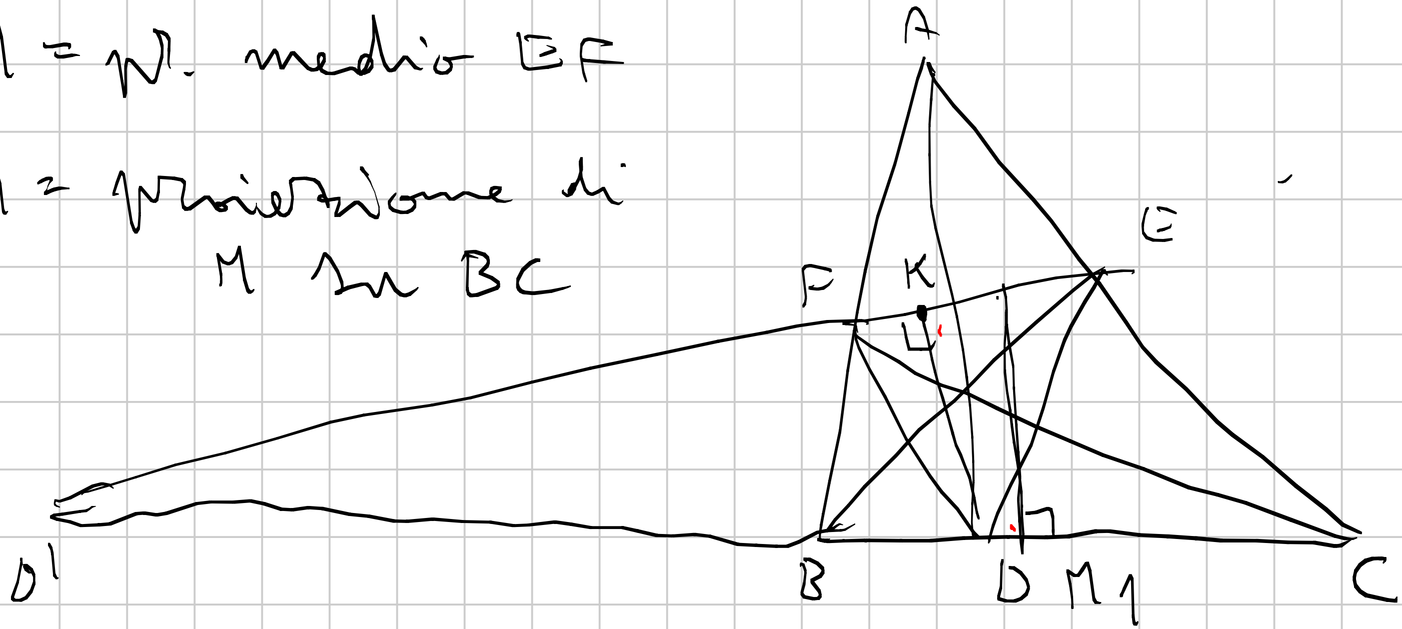
$$= \frac{1}{4}\left(a^2 + (b-c)^2 - 2a(b+c) \right)$$

$$= \frac{1}{4}\left(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca \right)$$

$\Rightarrow \overline{D'E'F'} \perp OI$ per il LEMMA

M = pt. medio EF

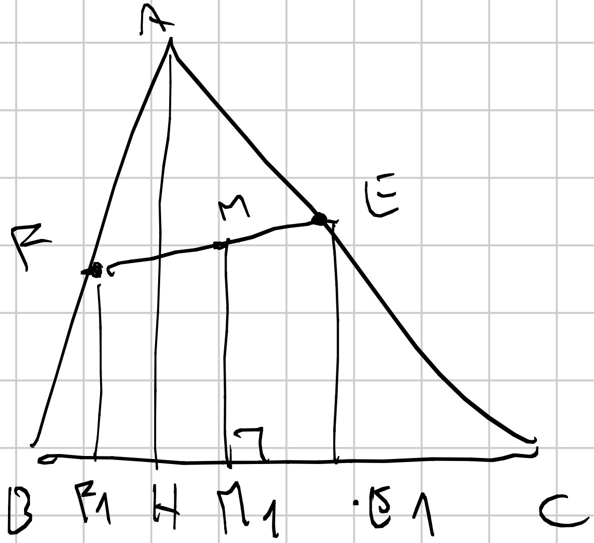
M_1 = proiezione di M su BC



DM_1MK ciclico

$$\Rightarrow D'K \cdot D'M = D'D \cdot D'M_1 = \text{pow}_{C(N)}(D')$$

$$D'D = BD' + BD = \frac{2abc}{(b+c)(b-c)}$$



$$BM_1 = \frac{BF_1 + BE_1}{2} = \frac{BF_1 + (BC - CE_1)}{2}$$

$$BF_1 = BH \cdot \frac{a}{a+b}$$

perché $\frac{BF_1}{BH} = \frac{BF}{BA} = \frac{a}{a+b}$

$$BH = c \cdot \cos\beta = c \cdot \frac{a^2+c^2-b^2}{2ac}$$

$$\Rightarrow \quad BF_1 = \frac{a^2+c^2-b^2}{2(a+b)}$$

$$BM_1 = \frac{1}{2}\left(\underbrace{\frac{2a(a+c)(a+b)}{2(a+c)(a+b)}}_{a} + \underbrace{\frac{(a+c)(a^2+c^2-b^2)-(a+b)(a^2+b^2-c^2)}{2(a+c)(a+b)}}_{\sim\ BF_1 - CE_1} \right)$$

$$= \frac{1}{2}\left(\sim\sim\sim + \frac{(b-c)(a+b+c)^2}{2(a+c)(a+b)} \right)$$

$$\Rightarrow D'M_1 = D'B + BM_1 =$$

$$\frac{1}{2}\left(\frac{2a(a+c)(a+b)}{2(a+c)(a+b)} + \frac{(b-c)(a+b+c)^2}{2(a+c)(a+b)} \right) + \frac{ac}{b-c}$$

$$D'M_1 \cdot D'D = \frac{abc\left((b-c)\left(2a(a+c)(a+b) + (b-c)(a+b+c)^2\right)\right)}{2(a+c)(a+b)(b+c)(b-c)^2}$$

$$+ \frac{abc}{\sim\!\sim}\left(4ac(a+c)(a+b)\right)$$

$$= \frac{abc}{\sim\!\sim}\left(2a(c+b)(a+c)(a+b) + (b-c)^2(a+b+c)^2\right)$$

$$\mathrm{Pow}_{((N))}(D') - \mathrm{Pow}_{((O))}(D') =$$

$$= \frac{abc}{2(a+c)(a+b)(b+c)(b-c)^2}\left[\text{\textasciitilde\textasciitilde} - 2a(a+c)(a+b)(b+c)\right]$$

$$= \frac{abc(a+b+c)^2}{2(a+c)(a+b)(b+c)}$$

$\Rightarrow \quad \overline{D'E'F'} \neq ON \quad \Rightarrow O, I, N$ allineati.