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ALGEBRA

Note Title 27/01/2017

A4. $f:\mathbb{R}\to\mathbb{R}$ $\quad \forall x,y\in\mathbb{R}$

$$f(x+y^2)\geq (y+1)f(x)$$

① $f(x)=e^x$ $\qquad e^{y^2}\geq y+1 \quad \forall y$ NO!!!

Bozza $\qquad \dfrac{f(x+\varepsilon)}{f(x)} \geq$ mostro che cresce troppo

$f(x)\geq 0$ e $f(x)\leq 0$

$y=-1 \qquad f(x+1)\geq 0$ ✓

$x=x+y^2 \qquad f(x+2y^2)\geq (y+1)f(x+y^2)$
$\geq (y+1)^2 f(x)$

$$f(x+ny^2)\geq (y+1)^n f(x)$$

$y=\frac{1}{n} \qquad f\left(x+\frac{1}{n}\right)\geq \left(\frac{1}{n}+1\right)^n f(x)$

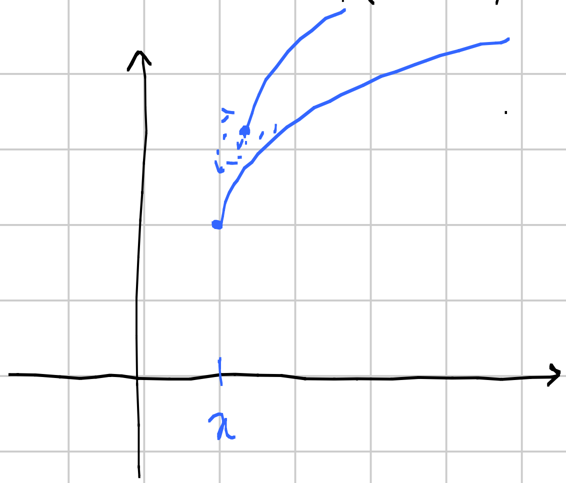
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

$x=x+\frac{1}{n} \qquad f\left(x+\frac{1}{n}+\frac{1}{n}\right)\geq\left(\frac{1}{n}+1\right)^n f\left(x+\frac{1}{n}\right)\geq\left(\frac{1}{n}+1\right)^{2n}f(x)$

$$f(x+1)\geq\left(1+\frac{1}{n}\right)^{n^2}f(x) \qquad \forall x$$

$$\underline{f(x)} \le \frac{f(x+1)}{\left(1+\frac{1}{n}\right)^{n^2}} \;\underline{\sim}\; \frac{f(x+1)}{e^{n}} = 0$$

Altra idea: $f(x+z) \ge (1+\sqrt{z})f(x)$ $\quad z>0$



$z = \varepsilon$

$f(x+n\varepsilon) \ge \dots$ grande

A5. $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \le$

$$4 + \frac{(ab+ac+ad+bc+bd+cd)^2}{3abcd}$$

① $\iff (ab+cd-ac-bd)^2 + (ac+bd-ad-cb)^2$

$+(ad+bc-ab-cd)^2 \ge 0$

② $a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2+6abcd$

$\ge \sum_c a^2(bc+cd+db)$

$x^2+y^2+z^2 \ge 0 \qquad x^2+y^2+z^2 \ge xy+yz+zx$

$x = ab+cd, \quad y = ac+bd, \quad z = ad+bc$

(3) $S = b+c+d$, $Q = bc + cd + db$

$P = bcd$

$$(a+S)\left(\frac{1}{a}+\frac{Q}{P}\right) \le 4 + \frac{(aS+Q)^2}{3aP}$$

$$\Leftrightarrow \boxed{(S^2-3Q)}\,a + \frac{\boxed{Q^2-3PS}}{a} \ge SQ - 9P$$

$$LHS \overset{\checkmark}{\ge} 2\sqrt{(S^2-3Q)(Q^2-3PS)} \overset{?}{\ge} SQ - 9P$$

6. $f: \mathbb{R} \to \mathbb{R}$ $\quad x, y \in \mathbb{R}$, $n \geq 2$

$$f(x^n + 2f(y)) = (f(x))^n + y + f(y)$$

1) f è iniettiva $\qquad g(f(y), x) = y$

y_1, y_2 con $f(y_1) = f(y_2)$

2) f è surgettiva $\qquad f(g(x,y)) = \underset{\substack{\uparrow \\ \text{surgettiva}}}{h(y)}$

n dispari $\quad x$ t.c. $\quad x^n + 2f(y) = y$

$f(x) = (-y)^{\frac{1}{n}}$

3) $\exists! \, \alpha \in \mathbb{R} : f(\alpha) = 0$ $\qquad y \leftarrow \alpha$

$$f(x^n) = (f(x))^n + \alpha \qquad \forall n \geq 2$$

$$f(x^n + 2f(y)) = f(x^n) - \alpha + y + f(y)$$

n dispari x^n invertibile

$$f(x + 2f(y)) = f(x) - \alpha + y + f(y) \qquad z = 2f(y) \quad y = f^{-1}\left(\frac{z}{2}\right)$$

$$f(x+z) = f(x) + g(z)$$

4) Simmetrizzo $f(x) + g(z) = f(x+z) = f(z) + g(x)$

$$g(x) = f(x) + \underbrace{g(z) - f(z)}_{z \text{ cost}}$$

$$g(x) = f(x) + c$$

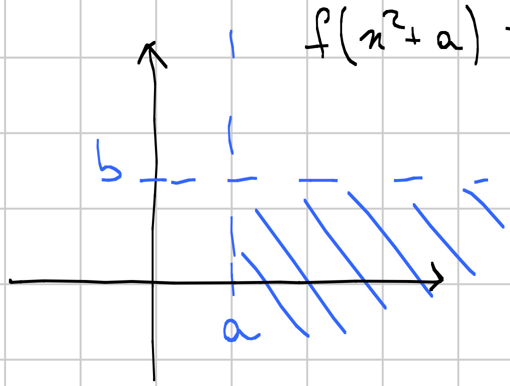
$$f(x+z) = f(x) + f(z) + c$$

$$f(x+z) + c = f(x) + c + f(z) + c$$

$f(x)+c$ soddisfa eq di Cauchy

5) $n = 2$ $\qquad f(x^2 + 2f(y)) = f(x)^2 + y + f(y) \geq y + f(y)$

$$f(x^2 + a) \geq b$$



$$f(x) = ax + b$$

6) Sostituisco e verifico

$a = 1 \quad b = 0$

Considerazioni

$n=2 \qquad (x,y) \qquad (-x,y)$

$$f(x)^2 = f(-x)^2 \qquad f(-x) = -f(x)$$

$$f(x^n) = [f(x)]^n$$

$$f(x+2f(y)) = f(x) + \boxed{y + f(y)} = f(x) + f(2f(y))$$

$x=0 \qquad f(2f(y)) = y + f(y)$

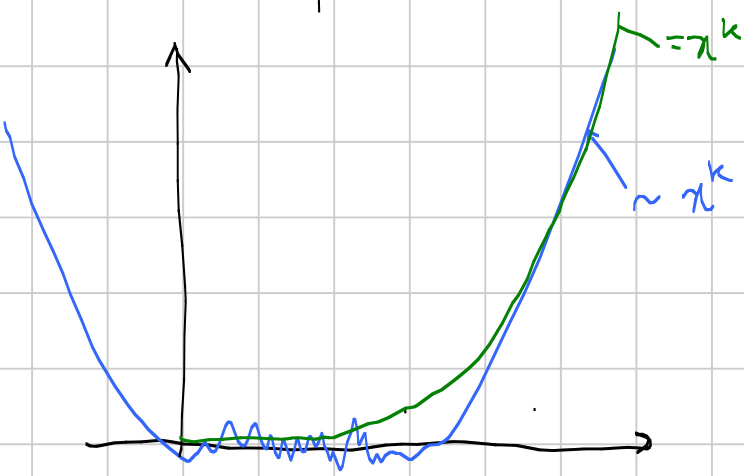
$$\boxed{f(\alpha)=0 \quad \text{con} \quad \alpha \neq 0 \implies f(-\alpha)=0}$$

$$f(x+z) = f(x) + f(z)$$

A7. $p(x) \in \mathbb{Z}[x] \qquad p(x) = \sum_{i=0}^{k} a_i x^i \qquad a_k = 1$

$\forall n \geq 1 \quad \exists x_n \in \mathbb{Z} \quad p(x_n) = 2^n$

1) Se $k=1 \quad p(x) = x+a \quad$ va bene $\quad x_n = 2^n - a$



$q(x) = x^k$

$x = 2^m$

$q(2^m) = 2^{mk}$

$k \mid n$ va bene

Traslo orizzontalmente $p(x)$ per avvicinarlo a x^k

$$p(x\pm 1) = \sum_{i=1}^{k} a_i (x\pm 1)^i = (x\pm 1)^k + a_{k-1}(x\pm 1)^{k-1} + \dots$$

$$= x^k + \underbrace{(a_{k-1} \pm k)}_{\gtrless 0} x^{k-1} + q^{\pm}(x)$$

$g \leq k-2$

wlog : $|a_{k-1}| < k$

$$p(x-1) = x^k - c_1 x^{k-1} + q^-(x)$$
$$p(x+1) = x + c_2 x^{k-1} + q^+(x)$$

$p(x-1) < x^k < p(x+1)$ per x molto grande $x > 1$ e

$x = 2^m$ $\quad x c_1 > (k-1) \max |\text{coeff } q^-|$

$p(2^m - 1) < 2^{mk} < p(2^m + 1)$ $\quad x c_2 > (k-1) \max |\text{coeff } q^+|$

$p(x_{mk}) = 2^{mk}$

$$\Rightarrow p(2^m) = 2^{mk} = \sum_{i=0}^{k} a_i (2^{mi}) = 2^{mk} + r(2^m)$$

$r(2^m) = 0 \quad \forall m \quad r$ polinomio $\quad r \equiv 0$

$p(x) = x^k$ (traslato)

(attenzione al punto delicato se k è pari!)

$k | n \quad \forall n \Rightarrow k = 1 \quad p(x)$ traslato $= x$

$p(x) \in \mathbb{Z}[x] \quad \forall m, n \in \mathbb{Z} \quad m - n \mid p(m) - p(n)$

x_i, $\quad x_1 = 0$

$a_n - a_{n+1} \mid 2^n$ $\qquad \Delta_n = a_{n+1} - a_n$

$a_{n+1} - a_{n+2} \mid 2^{n+1}$

$a_n - a_{n+2} \mid 3 \cdot 2^n$

$$\Delta_n = \begin{cases} 2\Delta_{n-1} \\ -4\Delta_{n-1} \end{cases}$$

$$|\Delta_n| \geq |a_n|$$

$$|a_n| \geq 2^{n-2} |a_2| \qquad a_2 = \Delta_1$$

$\deg p \geq 2$ $\qquad p(a_n) \geq a_2^2 \left(2^{2(n-2)}\right)$