WC 2018

ALGEBRA

Note Title 25/01/2018

$\boxed{A4}$ $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{x f(y)}\right) = f(y) \qquad \forall x,y>0$$

$$\frac{y}{f(x+1)} \overset{?}{=} \frac{x+1}{x f(y)} \qquad \begin{cases} x+1 = y \\ y = \frac{x+1}{x} \end{cases} \qquad \begin{cases} x=1 \\ y=2 \end{cases} \qquad f\left(\frac{2}{f(2)}\right) = \frac{1}{2} f(2)$$

$$\frac{2}{f(2)} \xrightarrow{f} \frac{f(2)}{2}$$

$$\frac{x+1}{x f(y)} \neq y \qquad \forall x,y$$

$$1+\frac{1}{x} = \frac{x+1}{x} \neq y f(y) \quad \Rightarrow \quad \boxed{y f(y) \leq 1 \quad \forall y}$$

uguaglianze per $y = \frac{2}{f(2)}$

$$f(y) \leq \frac{1}{y}$$

$$\boxed{f(y) = f(A) + f(B) \leq \frac{1}{A} + \frac{1}{B} = \frac{f(x+1)}{y} + \frac{x f(y)}{x+1}} \leq \frac{1}{y(x+1)} + \frac{x}{y(x+1)} = \frac{1}{y}$$

(S. Di Marino)

$y \leftarrow \frac{2}{f(2)}$ tutte uguaglianze

$$\frac{2}{f(2)} = 4$$

$$f(A) = \frac{1}{A} = \frac{f(x+1)}{y} \qquad f(B) = \frac{1}{B} = \frac{x f(y)}{x+1} \qquad f(x+1) = \frac{1}{x+1} \quad \forall x$$

(non si chiude)

$$\frac{f(y)}{x+1} \leq \frac{f(x+1)}{y} \iff y f(y) \leq (x+1) f(x+1) \qquad \forall x,y$$

$$\Rightarrow \boxed{y f(y) = k \qquad \forall y>1}$$

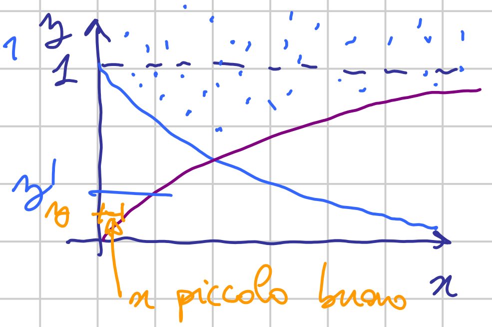
$$\frac{k}{y} = f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{x f(y)}\right) = k\,\frac{f(x+1)}{y} + k\,\frac{x f(y)}{x+1} = k^2 \cdot \frac{1}{y} \qquad \forall y>1$$

$y>1$

$$\frac{y}{f(x+1)} \geq y(x+1) \overset{!}{>} 1$$

$$\frac{x+1}{x f(y)} \geq \frac{(x+1)y}{x} \overset{!}{>} 1$$

$$\boxed{k=1}$$



$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{x f(y)}\right) = f\big(y(x+1)\big) + \frac{x}{x+1} f(y)$$

piccolo

sopra visto

$$f(y) = f\big(\underbrace{y(x+1)}_{y'>y}\big)(x+1) = f(y')(x+1) = f\big(y'(x+1)\big)(x+1)^2 = f\big(y(x+1)^2\big)(x+1)^2$$

$\forall x < \bar{x}(y)$

$$= \dots f\big(y(x+1)^n\big)(x+1)^n$$

$$= \frac{1}{y(x+1)^n}(x+1)^n = \frac{1}{y}$$

$\boxed{A5}$

$\sum_{cyc} \frac{a}{b^3+4} \geq ?$ $\qquad \sum_{cyc} a = 4 \qquad a \geq 0$

$a = b = c = d = 1 \to \frac{4}{5}$

$\boxed{a = b = 2 \quad c = d = 0 \to \frac{2}{3}}$

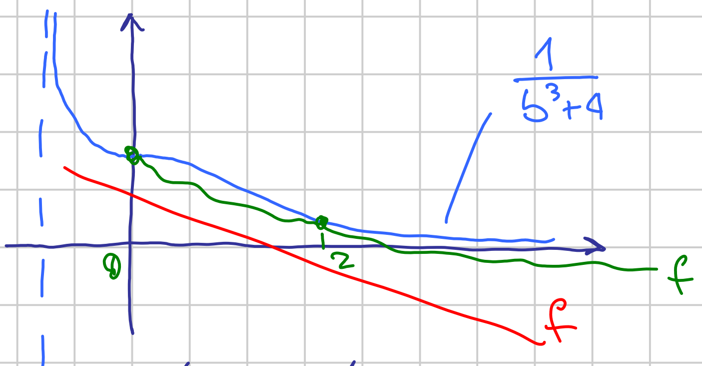
$c = d = 0 \quad a + b = 4 \dashrightarrow a = 2, b = 2$ minimo

$\sum_{cyc} \frac{a}{b^3+4} \geq \frac{2}{3}$ $\qquad$ ~~Titu's Lemma~~ $\emptyset$ non è sharp su 0,2

Dovrò stimare i singoli addendi

(Strada I) $\qquad \frac{a}{b^3+4} \geq a f(b)$

più semplice



$$f(x) = \frac{1}{4} - \frac{x}{12}$$

$$\frac{1}{x^3+4} \geq \frac{1}{4} - \frac{x}{12} \iff 12 \geq (x^3+4)(3-x) = 3x^3 - x^4 + 12 - 4x$$

$$\iff 0 \geq x(x^3 - 3x^2 + 4) = -x(x-2)(x^2-x-2) = -x(x-2)^2(x+1) \quad \text{ok!}$$

$$\sum_{cyc} \frac{a}{b^3+4} \geq \sum_{cyc}\left(\frac{a}{4} - \frac{ab}{12}\right) = 1 - \sum_{cyc} \frac{ab}{12} \overset{?}{\geq} \frac{2}{3} \iff \sum_{cyc} ab \leq 4$$

(Strada II)

$$\frac{2}{3} \overset{?}{\leq} \sum_{cyc} \frac{a}{4+b^3} \leq \sum_{cyc} \frac{a}{4} = 1$$

$$1-\sum_{cyc}\frac{a}{4+b^3} = \sum_{cyc} a\left(\frac{1}{4} - \frac{1}{4+b^3}\right) = \frac{1}{4}\sum_{cyc}\frac{ab^3}{b^3+4} \overset{?}{\leq} \frac{1}{3}$$

$$\frac{ab^3}{b^3+4} \leq ?$$

$$b^3+4 = \frac{1}{2}b^3+\frac{1}{2}b^3+\frac{1}{2}2^3 \geq \frac{3}{2}\,2b^2 = 3b^2$$

$$\leq \frac{ab}{3} \qquad \frac{1}{4}\sum_{cyc}\frac{ab}{3} \leq \frac{1}{3}$$

Conclusione

$\sum_{cyc} ab$

$$4^2 = (a+b+c+d)^2 = \sum a^2 + 2\sum ab + 2(ac+bd)$$

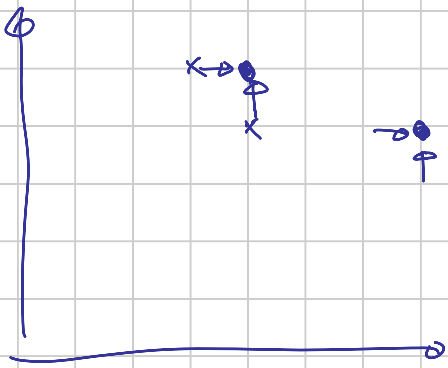
$$(a-b+c-d)^2 = \sum a^2 - 2\sum ab + 2(ac+bd)$$

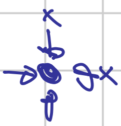
$$4^2 \geq 4^2 - (\quad)^2 = 4\sum_{cyc} ab$$

Fine

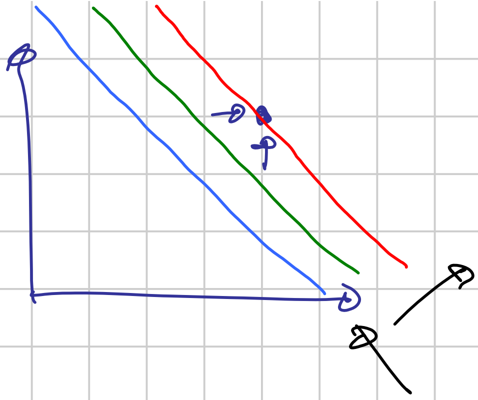
$$f: \mathbb{Z}^2 \to [0,1]$$

$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2}$$

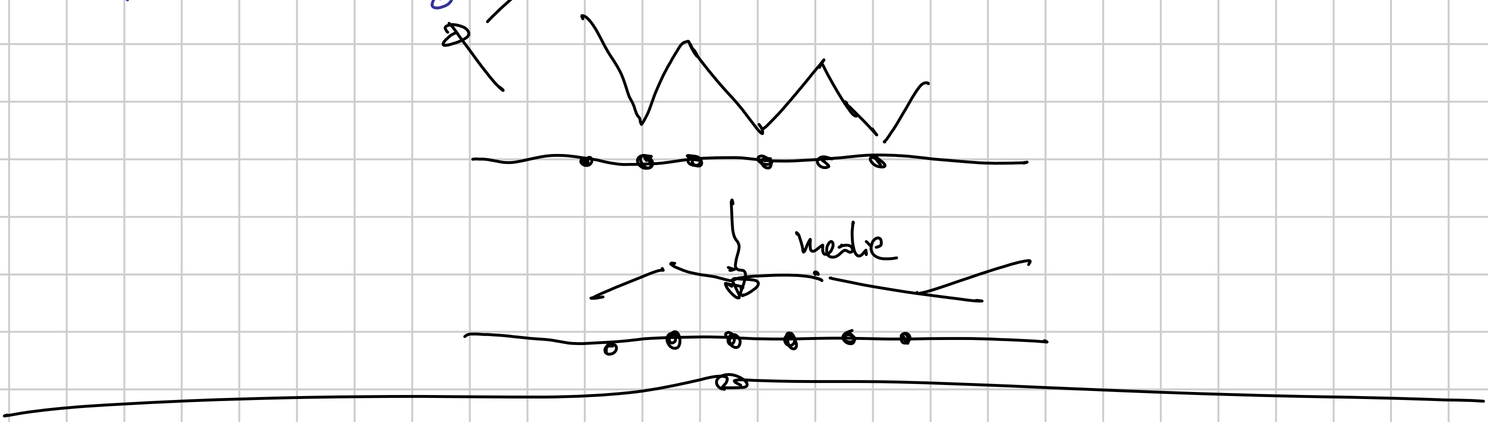


ES:  grafico in cui il valore di ogni punto è la media dei 4 intorno

se $\exists M$ t.c. $f(x,y) \leq M \ \forall x,y$, allora sono tutti uguali

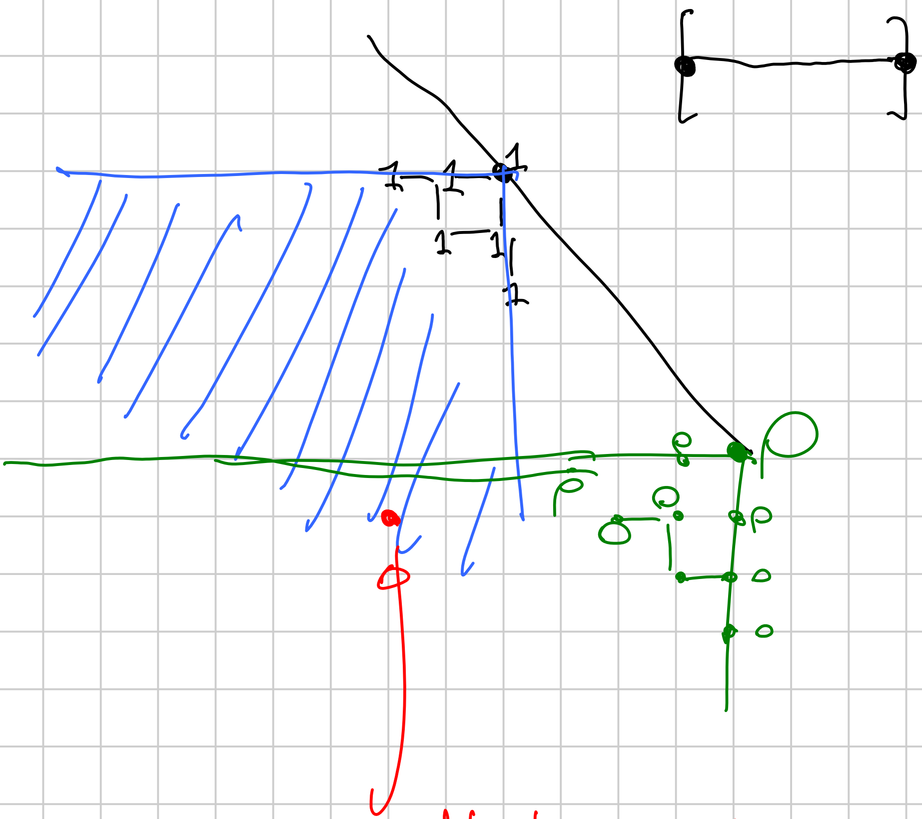


blu determina verde che determina rosso, facendo delle medie



$1 = \frac{1+1}{2}$ è l'unico modo di scrivere 1 come media di due valori $[0,1]$

$0 = \frac{0+0}{2}$ è l'unico modo -- 0

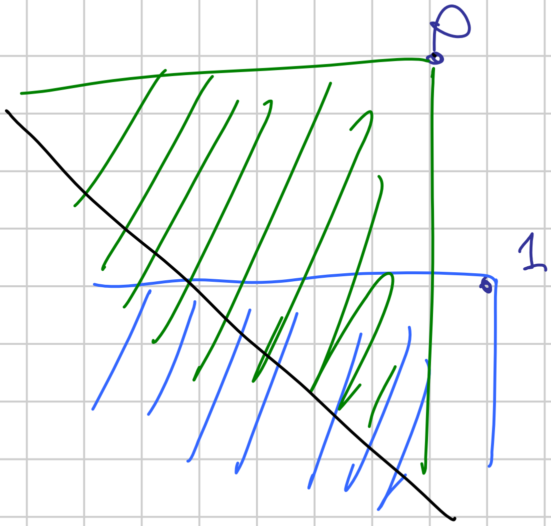


Se ho un 1 nella funzione, in basso e a sinistra devono esserci degli uni ⇒ ho un intero "quadrante" di uni

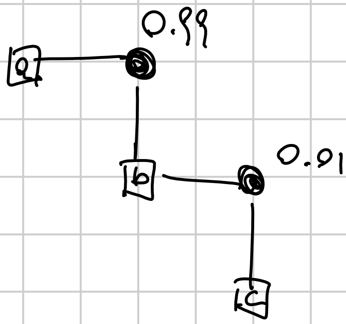
E la stessa cosa per 0

ogni punto nell'intersezione di questi quadranti non può essere sia 0 che 1 ⇒ assurdo!

⇒ Non posso avere sia 0 che 1 contemporaneamente



Posso ottenere adiacenti 0.99 e 0.01?



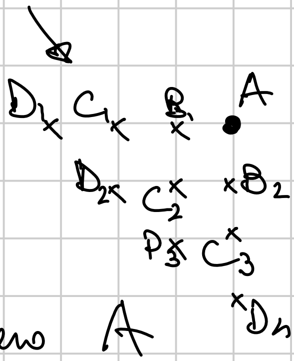
$$0.99 = \frac{a+b}{2} \le \frac{1+b}{2}$$

$$\Rightarrow b \ge 0.98$$

$$0.01 = \frac{b+c}{2} \ge \frac{b+0}{2}$$

$$\Rightarrow b \le 0.02$$

assurdo!



B_1, B_2 determinano A

C_1, C_2, C_3 determinano A

D_1, D_2, D_3, D_4 determinano A

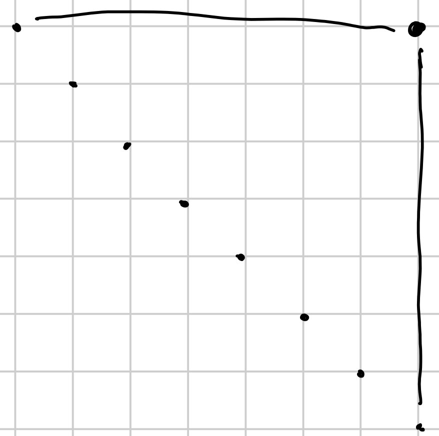
$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2} =$$

$$= \frac{\dfrac{f(x-2,y)+f(x-1,y-1)}{2} + \dfrac{f(x-1,y-1)+f(x,y-2)}{2}}{2}$$

$$= \frac{f(x-2,y)+2f(x-1,y-1)+f(x,y-2)}{4}$$

$\frac{1}{4}$ $\frac{2}{4}$ $\frac{1}{4}$

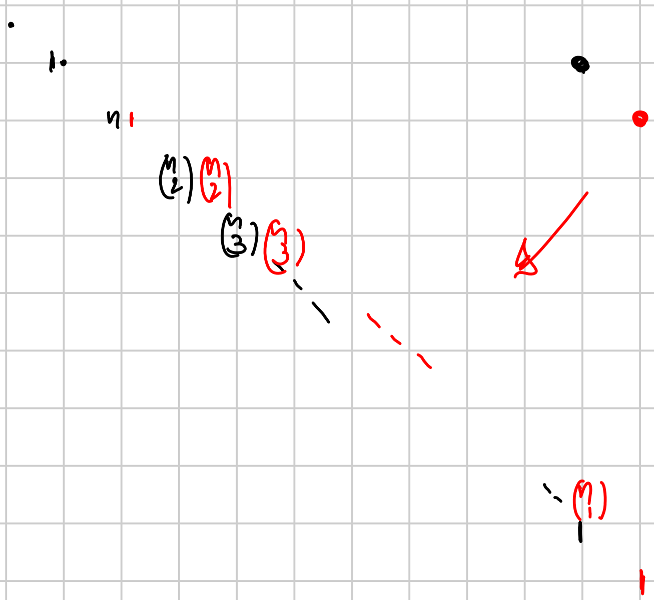
$$= \frac{\boxed{1}f(x-3,y)+\boxed{3}f(x-2,y-1)+\boxed{3}f(x-1,y-2)+\boxed{1}f(x,y-3)}{8}$$



(*)

$$= \frac{\binom{n}{0}f(x-n,y)+\binom{n}{1}f(x-n+1,y-1)+\cdots+\binom{n}{n-1}f(x-1,y-n+1)+\binom{n}{n}f(x,y-n)}{2^n}$$

$\frac{1}{2^n}$



Esempio:

$$f(x,y)\cdot 2^4$$

$$f(x-5,y+1) \quad f(x-4,y) \quad f(x-3,y-1) \quad f(x-2,y-2) \quad f(x-1,y-3) \quad f(x,y-4)$$

1 4 6 4 1

1 4 6 4 1

1 uguale 4 uguali 4 uguali 1 uguale

$$f(x-1,y+1)\cdot 2^4$$

10 termini uguali
6 diversi

Quanti sono i termini uguali

$$2^n\left[f(x,y)-f(x-1,y+1)\right]=\Big(f(x-n,y)+\binom{n}{1}f(x-n+1,y)+\dots+\binom{n}{n-1}f(x-1,y-n+1)+\binom{n}{n}$$

$$-f(x-n-1,y+1)-\binom{n}{1}f(x-n,y)-\dots-\binom{n}{n}f(x-1,y-n+1)\Big)$$

(semplificando termini uguali) $\boxed{n=2m}$

$= \binom{2m}{m}$ termini col segno $+$ $\quad \binom{2m}{m}$ termini col segno $-$

$$\leq \binom{2m}{m}\cdot 1-\binom{2m}{m}\cdot 0=\binom{2m}{m}$$

$$\Rightarrow \quad f(x,y)-f(x+1,y-1)\leq \frac{\binom{2m}{m}}{2^{2m}}$$

(rapporto tra il binomiale centrale e la somma di tutti i binomiali)

$$1 \quad 4 \quad 6 \quad 4 \quad 1 \qquad \frac{6}{16}$$

$$1 \quad 6 \quad 10 \quad 15 \quad 10 \quad 6 \quad 1 \quad \sim \quad \frac{10}{64}$$

È vero che $\dfrac{\binom{2n}{n}}{2^{2n}}$ tende a 0?

$$n! = \frac{n^n}{e^n}\cdot\sqrt{2\pi n}\cdot\left(1+o(1)\right)$$

$$\frac{1}{2^{2n}}\binom{2n}{n} \approx \frac{1}{2^{2n}}\frac{(2n)!}{n!\,n!} = \frac{1}{2^{2n}}\,\frac{\frac{(2n)^{2n}}{\sqrt{2\pi n}}\;e^{2n}}{\frac{n^n}{\sqrt{2\pi n}\,e^n}\;\frac{n^n}{\sqrt{2\pi n}\,e^n}} = \frac{1}{\sqrt{2\pi n}}$$

Si riesce anche a dimostrare direttamente che

$$\frac{1}{2^{2n}}\cdot\binom{2n}{n} \to 0 :$$

Idea:

$$\binom{2n}{n} = \frac{(2n)!}{n!\;n!} = \frac{2n\cdot(2n-1)}{n\cdot n}\cdot\frac{[2(n-1)]!}{(n-1)!\,(n-1)!} =$$

$$= \frac{2\cancel{n}(2n-1)}{n\cdot\cancel{n}}\binom{2(n-1)}{n-1} =$$

$$\binom{2n}{n}\cdot\frac{1}{2^{2n}} = \frac{2n-1}{2n}\cdot\binom{2(n-1)}{n-1}\cdot\frac{1}{2^{2(n-1)}} = \left(1-\frac{1}{2n}\right)\cdot\binom{2(n-1)}{n-1}\cdot\frac{1}{2^{2(n-1)}}$$

$$\frac{1}{2^{2n}}\binom{2n}{n} = \left(1-\frac{1}{2n}\right)\cdot\left(1-\frac{1}{2n-2}\right)\cdot\left(1-\frac{1}{2n-4}\right)\cdot \dots \cdot\left(1-\frac{1}{2}\right)$$

Idea: magari telescopizza!

Se per esempio fosse
$$\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\left(1-\frac{1}{n-2}\right)\cdot \dots \cdot\left(1-\frac{1}{2}\right) =$$
$$= \frac{\cancel{n-1}}{n}\cdot\frac{\cancel{n-2}}{\cancel{n-1}}\cdot\frac{\cancel{n-3}}{\cancel{n-2}}\cdot \dots \cdot\frac{\cancel{1}}{\cancel{2}} = \frac{1}{n}$$

non funziona =(

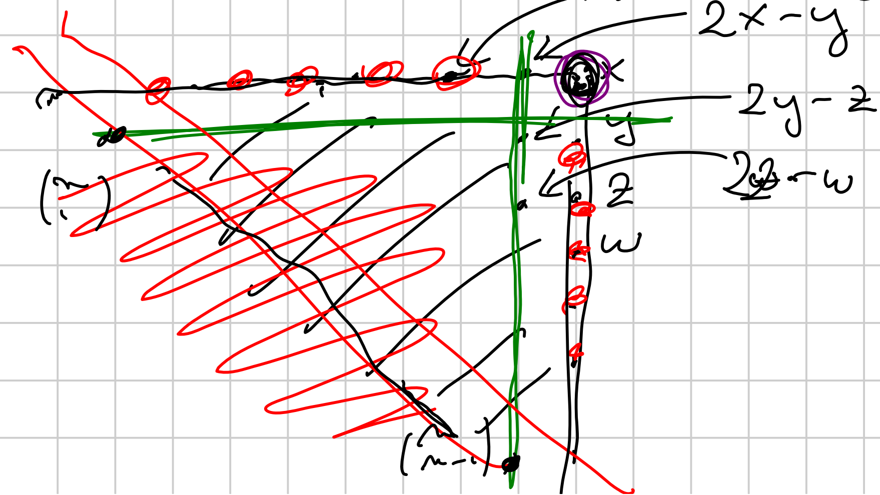
$$\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2 = \left(1-\frac{1}{2n}\right)^2\left(1-\frac{1}{2n-2}\right)^2\left(1-\frac{1}{2n-4}\right)^2 \dots \left(1-\frac{1}{2}\right)^2 \leq$$

$$\leq \left(1-\frac{1}{2n+1}\right)\left(1-\frac{1}{2n}\right)\left(1-\frac{1}{2n-1}\right)\left(1-\frac{1}{2n-2}\right)\dots \dots \dots \left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)$$

$$= (\text{telescopizza}) = \frac{1}{2n+1}$$

$$\frac{1}{2^{2n}}\binom{2n}{n} \leq \sqrt{\frac{1}{2n+1}}$$

$$4\overset{0}{x} - 2y - 2y + z = \overset{0}{4x} - 4y + z$$



$\cdot \geq 0$

$\cdot \geq 0$

$\cdot \geq 0$

$\cdot \geq 0$

$$(2x-y)+(2y-z)+(2z-w) = 2x+y+z-w \geq 0$$

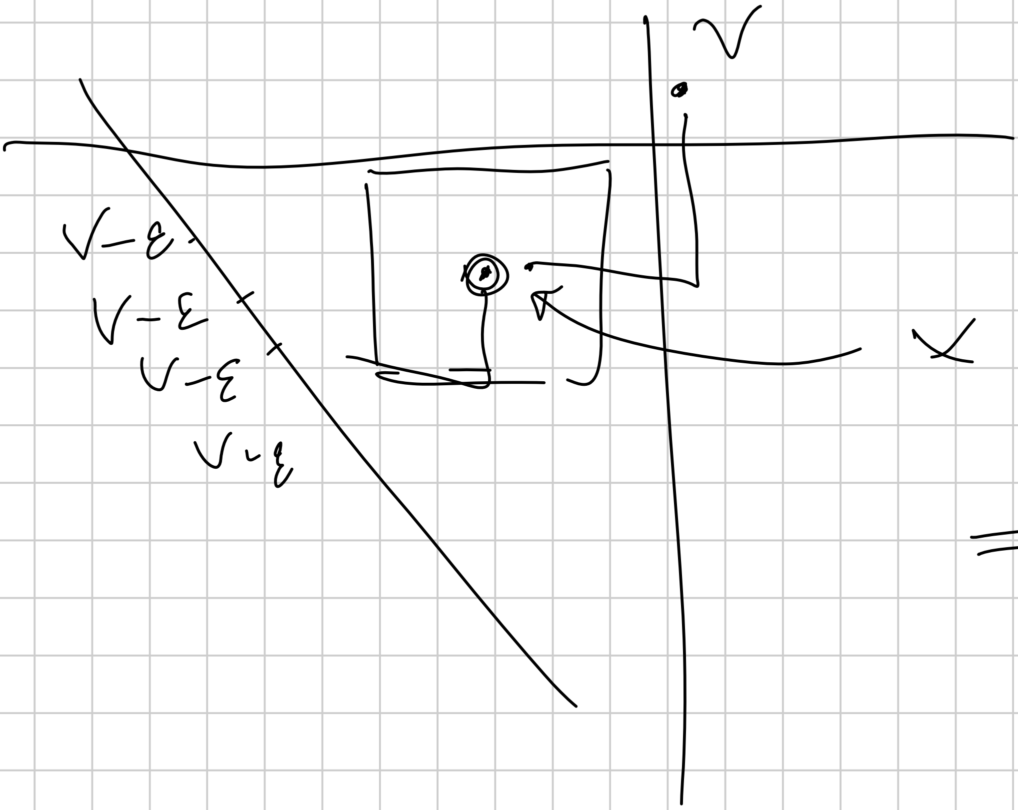
$\sup(\text{rossi}) = p$ $\qquad V$

$$2^n V \leq 2^n p - 2np + 2n$$

Claim: $\quad p = V \qquad p = V - \varepsilon \quad \forall \varepsilon$

$$\forall \varepsilon \ \exists n : \quad p = V - \varepsilon$$

$$\frac{V-p}{1-p} \leq \boxed{\frac{2n}{2^n}} \to 0$$



$V \geq x$

$x \geq V - \varepsilon$

$\Rightarrow \quad x = V$