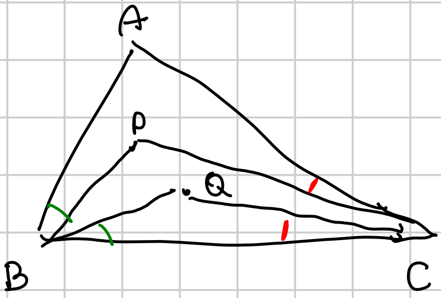
WC 2018 – Geometria Contosa

Note Title 25/01/2018

G8



Hp. P, Q

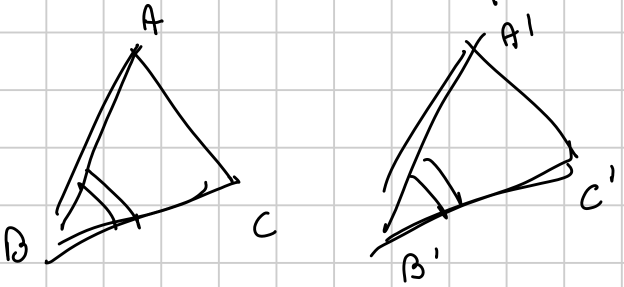
$\measuredangle ABP = \measuredangle QBC$

$\measuredangle ACP = \measuredangle QCB$

Th. Mostrare che esiste un punto X sulla cfr. circoscritta t.c. $XPI \underset{dir.}{\sim} XIQ$

[Complessi]

Sol. Com'è fatto "un" ("il") punto X che soddisfi $XPI \underset{dir.}{\sim} XIQ$



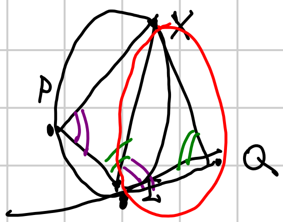
$ABC \simeq A'B'C'$

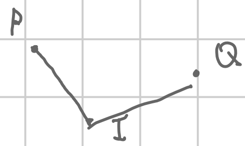
$$\frac{a-b}{c-b} = \frac{a'-b'}{c'-b'}$$

$$XPI \underset{dir.}{\sim} XIQ \iff \frac{p-x}{i-x} = \frac{i-x}{q-x} \iff x^2 - (p+q)x + pq = x^2 - 2ix + i^2$$

$$\iff x = \frac{pq - i^2}{p+q-2i}$$

Un solo punto



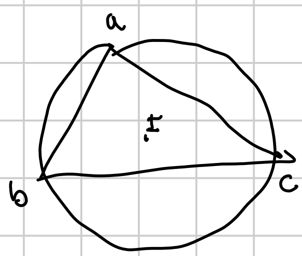


Th del problema: p e q come nel testo $\Rightarrow x\bar{x} = 1$

$\Downarrow$

$\odot ABC$ = cfr unitaria

Fatto (u, v, w)



Lemma: $\exists^{mo}$ u, v, w t.c.

$$\begin{cases} u^2 = a \\ v^2 = b \\ w^2 = c \\ -(uv + vw + uw) = i \end{cases}$$

Setting: $\odot ABC \equiv$ cfr. unitaria.

Tornando al prob: Vogliamo che

$$\frac{pq - i^2}{p+q-2i} \cdot \frac{\bar{p}\bar{q} - \bar{i}^2}{\bar{p}+\bar{q} - 2\bar{i}} = 1$$

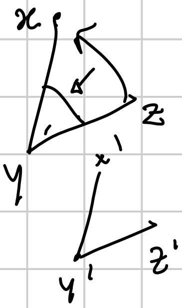
Notare che $\boxed{\bar{i} = -\frac{u+v+w}{uvw}}$ perché $\bar{u} = \frac{1}{u}$ e analoghe.

Problema: Scrivere $\boxed{\bar{p}\bar{q}}$ e $\boxed{\bar{p}+\bar{q}}$ in termini del resto

Scriviamo le ipotesi

$\measuredangle ABP = \measuredangle QBC$

$$\frac{\frac{p-b}{a-b}}{\frac{\bar p-\bar b}{\bar a-\bar b}} = \frac{\frac{c-b}{q-b}}{\frac{\bar c-\bar b}{\bar q-\bar b}}$$



$$\frac{\frac{x-y}{z-y}}{\frac{\bar x-\bar y}{\bar z-\bar y}} = \frac{\frac{x'-y'}{z'-y'}}{\frac{\bar x'-\bar y'}{\bar z'-\bar y'}}$$

$\downarrow (u,v,w)$

$$\frac{\frac{p-v^2}{u^2-v^2}}{\frac{\bar p-\frac{1}{v^2}}{\frac{1}{u^2}-\frac{1}{v^2}}} = \frac{\frac{w^2-v^2}{q-v^2}}{\frac{\frac{1}{w^2}-\frac{1}{v^2}}{\bar q-\frac{1}{v^2}}} \Longrightarrow \frac{p-v^2}{u^2(1-\bar p v^2)} = \frac{w^2(1-\bar q v^2)}{(q-v^2)}$$

Com'è $\measuredangle ACP = \measuredangle QCB \Longrightarrow \frac{p-w^2}{u^2(1-\bar p w^2)} = \frac{v^2(1-\bar q w^2)}{(q-w^2)}$

Quindi

$$\begin{cases} \frac{(p-v^2)(q-v^2)}{u^2w^2} - 1 = -(\bar p+\bar q)v^2 + \bar p\bar q v^4 \\ \frac{(p-w^2)(q-w^2)}{u^2v^2} - 1 = -(\bar p+\bar q)w^2 + \bar p\bar q w^4 \end{cases}$$

$$\boxed{\bar p+\bar q} = \frac{w^{2}\left[\frac{(p-v^2)(q-v^2)-u^2w^2}{u^2w^2}\right] - v^{2}\left[\frac{(p-w^2)(q-w^2)-u^2v^2}{u^2v^2}\right]}{w^2v^4 - v^2w^4} =$$

$$= \frac{w^2pq - \cancel{v^2w^2(p+q)} + w^2v^4 - u^2w^4 - v^2pq + \cancel{v^2w^2(p+q)} - v^2w^4 + u^2v^4}{u^2w^2v^2(v^2-w^2)}$$

$$= \frac{u^2v^2+v^2w^2+u^2w^2-pq}{u^2w^2v^2}$$

$$\bar p\bar q = \frac{u^2+v^2+w^2-(p+q)}{u^2w^2v^2}$$

Cos'è che dovremo mostrare?

$$\left(pq-(uv+vw+uw)^2\right)\left(\frac{u^2+v^2+w^2-(p+q)}{u^2w^2v^2} - \frac{(u+v+w)^2}{u^2w^2v^2}\right) \overset{?}{=} (!)$$

(II)

$$\left(p+q+2(uv+vw+uw)\right) - \left(\frac{u^2v^2+v^2w^2+u^2w^2-pq}{u^2w^2v^2} + 2\frac{(u+v+w)uvw}{u^2v^2w^2}\right)$$

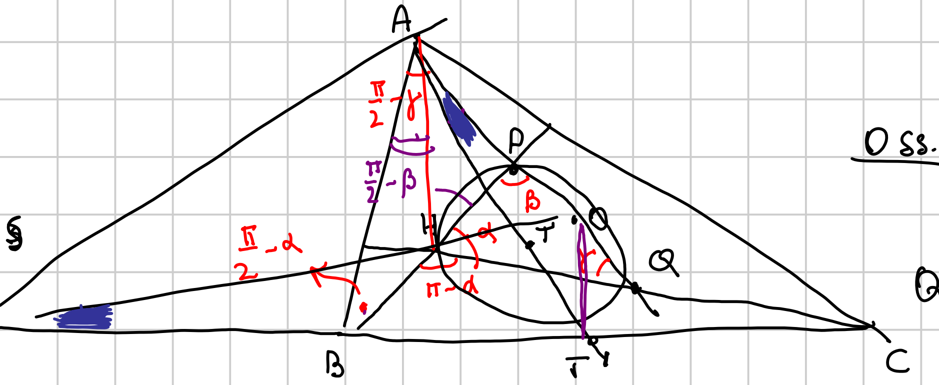
(IV)

Ok perchè (II) $= \frac{-2(uv+vw+uw)-(p+q)}{u^2v^2w^2}$

(IV) $= \frac{(uv+vw+uw)^2-pq}{u^2v^2w^2}$

✓

(G7)



Oss.1 $\overset{\triangle}{HPQ} \sim \overset{\triangle}{ABC}$

$P\hat{H}Q = \alpha$

$H\hat{P}Q = \beta$ $\rightsquigarrow$ $H\hat{Q}P = \gamma$

Oss.2 AH tange $\odot HPQ$

Infatti $A\hat{H}P = \gamma = P\hat{Q}H$

Quindi sia $S \doteq AA \cap BC$

sia $T \doteq$ Circocentro di HPQ

Per similitudine

$\measuredangle TAO = \measuredangle OSB \longrightarrow$

Sia $T' \doteq AT \cap BC \longrightarrow ASTO$ ciclico

Ma allora $OT'S = \pi - OAS = \frac{\pi}{2} \Rightarrow T'$ è il p.to medio di BC.

Soluzione Swag

$$S_A = \frac{b^2+c^2-a^2}{2}$$

$PH \perp AC$

$AO: b^2 S_B z = c^2 S_C y$

$H = (S_B S_C, S_A S_C, S_A S_B)$

$BH: x S_A = z S_C$

$P = (c^2 S_C^2, b^2 S_A S_B, c^2 S_A S_C)$

$$c^2 S_C^2 + b^2 S_A S_B + c^2 S_A S_C = b^2(S_A S_B + c^2 S_C) = b^2 \sum_{cyc} S_A S_B$$

$$D = (\underset{x}{S_C(b^2 S_B + c^2 S_C)}, \underset{y}{b^2 a^2 S_A}, \underset{z}{S_A(b^2 S_B + c^2 S_C)})$$

$\infty_{AC} = (\quad 1 \quad, \quad 0 \quad, \quad -1 \quad)$

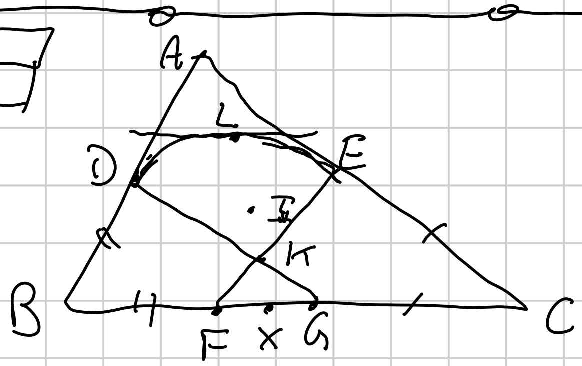
$\ell_P: -x a^2 \cancel{b^2} S_A + y \cancel{b^2}(b^2 S_B + c^2 S_C) - a^2 \cancel{b^2} S_A z = 0$

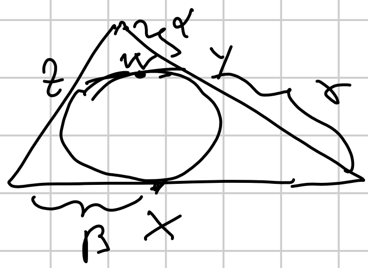
$\ell_Q: -x a^2 S_A - y a^2 S_A + z(b^2 S_B + c^2 S_C) = 0$

$AM: y - z = 0$

Se $R \in \ell_P \cap \ell_Q \Rightarrow y = z \Rightarrow R \in AM$.

Problema 5





Baricentriche su ABC

$t = XF \quad F = (0, CF, BF) = (0, \gamma+t, \beta-t)$

$BD = \beta - t$

$AD = \alpha+\beta-(\beta-t) = \alpha+t$

$AE = \alpha+t$

$CE = \alpha+\gamma-(\alpha+t) = \gamma-t$

$CG = \gamma - t$

$XG = \gamma-(\gamma-t) = t$

$G = (0, \gamma-t, \beta+t)$

$D = (\beta-t, \alpha+t, 0)$

$E = (\gamma-t, 0, \alpha+t)$

$$DG:\ \det\begin{pmatrix} x & y & z \\ \beta-t & \alpha+t & 0 \\ 0 & \gamma-t & \beta+t \end{pmatrix} = 0$$

$\gamma+t$ $\quad DG:\ x(\alpha+t)(\beta+t) - y(\beta^2-t^2) + z(\beta-t)(\gamma-t) = 0$

$\beta-t$ $\quad EF:\ x(\alpha+t)(\gamma+t) + y(\beta-t)(\gamma-t) - z(\gamma^2-t^2) = 0$

$$x(\alpha+t)(\gamma+t)2\beta + y(\beta-t)\left[(\beta-t)(\gamma-t)-(\beta+t)(\gamma+t)\right] = 0$$

$$x(\alpha+t)(\gamma+t)\beta = y(\beta-t)t(\beta+\gamma)$$

$$x(\alpha+t)(\beta+t)\gamma = z(\gamma-t)t(\beta+\gamma)$$

$$K = \left(t(\beta+\gamma)(\beta-t)(\gamma-t),\ (\gamma^2-t^2)\beta(\alpha+t),\ (\beta^2-t^2)\gamma(\alpha+t)\right)$$

$$K = \left(\frac{t}{\alpha+t}(\beta+\gamma),\ \frac{\gamma+t}{\beta-t}\beta,\ \frac{\beta+t}{\gamma-t}\gamma\right)$$

$X = (0, \gamma, \beta)$

$W = 2I - X$

$L = -\frac{t}{\alpha}A + \left(1+\frac{t}{\alpha}\right)(2I - X)$

L, I, K allineati $\Leftrightarrow I, K, \dfrac{L - 2\left(1+\frac{t}{\alpha}\right)I}{-\left(1+2\frac{t}{\alpha}\right)} = P$ allineati

$$P = \frac{-\frac{t}{\alpha}A - \left(1+\frac{t}{\alpha}\right)X}{-(1+2t/\alpha)} = \left(-\frac{t}{\alpha},\ -\left(1+\frac{t}{\alpha}\right)\frac{\gamma}{\beta+\gamma},\ -\left(1+\frac{t}{\alpha}\right)\frac{\beta}{\beta+\gamma}\right)$$

$P = (t(\beta+\gamma), (\alpha+t)\gamma, (\alpha+t)\beta)$

$I = (a, b, c) = (\beta+\gamma, \alpha+\gamma, \alpha+\beta)$

$$\det\begin{pmatrix} \cancel{\beta+\gamma} & \alpha+\gamma & \alpha+\beta \\ t\cancel{(\beta+\gamma)} & \gamma(\alpha+t) & \beta(\alpha+t) \\ \frac{t}{\alpha+t}\cancel{(\beta+\gamma)} & \frac{(\gamma+t)\beta}{\beta-t} & \frac{(\beta+t)\gamma}{\gamma-t} \end{pmatrix} \overset{?}{=} 0$$

$$\det\begin{pmatrix} 1 & \alpha+\gamma & \alpha+\beta \\ t & \gamma(\alpha+t) & \beta(\alpha+t) \\ 0 & \dfrac{\beta(\gamma+t)-\gamma(\beta-t)}{\beta-t} & \dfrac{\gamma(\beta+t)-\beta(\gamma-t)}{\gamma-t} \end{pmatrix} \overset{?}{=} 0$$

$$\det\begin{pmatrix} 1 & \alpha+\gamma & \alpha+\beta \\ t & \gamma(\alpha+t) & \beta(\alpha+t) \\ 0 & \dfrac{t\,\cancel{(\beta+\gamma)}}{\beta-t} & \dfrac{t\,\cancel{(\beta+\gamma)}}{\gamma-t} \end{pmatrix} \overset{?}{=} 0$$

$$\det\begin{pmatrix} 1 & \alpha+\gamma & \alpha+\beta \\ t & \gamma(\alpha+t) & \beta(\alpha+t) \\ 0 & \gamma-t & \beta-t \end{pmatrix} \overset{?}{=} 0$$

$$\det\begin{pmatrix} 1 & \alpha+\gamma & \alpha+\beta \\ 0 & \alpha(\gamma-t) & \alpha(\beta-t) \\ 0 & \gamma-t & \beta-t \end{pmatrix} \overset{?}{=} 0$$

$\mathrm{II} = \alpha\,\mathrm{III} \Rightarrow$ fe 0