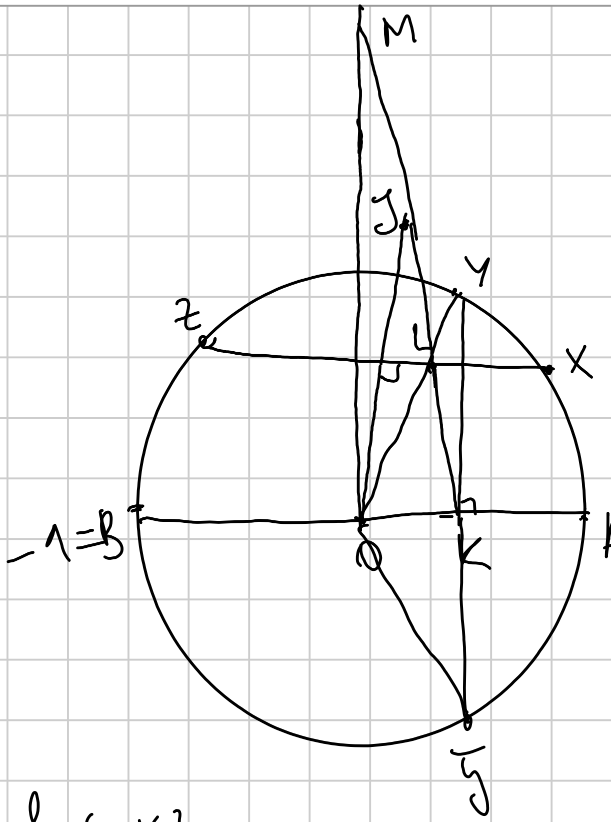
WC 2020 GEOMETRIA CONTOSA

Note Title 24/01/2020



Complessi!

$a, b, x, y, z \in$ circ. unitaria

$a\bar{a} = b\bar{b} = x\bar{x} = y\bar{y} = z\bar{z} = 1$

J simmetrico di O rispetto a XZ

$$\Rightarrow \frac{J+O}{2} = \frac{x+z}{2}$$

$$\Rightarrow J = x+z$$

$l \in xz$
$\in yo$

$$\begin{cases} \dfrac{x-l}{z-l} = \dfrac{\bar{x}-\bar{l}}{\bar{z}-\bar{l}} & \text{① condizione di allineamento} \\[2ex] \dfrac{y-l}{0-l} = \dfrac{\bar{y}-\bar{l}}{\bar{0}-\bar{l}} & \text{②} \end{cases}$$

② $-\bar{l}(y-l) = -l(\bar{y}-\bar{l}) = -l\left(\frac{1}{y}-\bar{l}\right)$

$+\cancel{\bar{l}l} - \bar{l}y = -\frac{l}{y} + \cancel{l\bar{l}} \Rightarrow \bar{l} = l/y^2$

① $(x-l)\left(\frac{1}{z}-\frac{l}{y^2}\right) = \left(\frac{1}{x}-\frac{l}{y^2}\right)(z-l)$

$$\frac{x}{z} - \frac{x}{y^2}l - \frac{l}{z} + \cancel{\frac{l^2}{y^2}} = \frac{z}{x} - \frac{l}{x} - \frac{lz}{y^2} + \cancel{\frac{l^2}{y^2}}$$

• xzy^2

$$(x+z)\cancel{(x-z)}y^2 = l(xy^2 + x^2z - zy^2 - z^2x)$$

$$(y^2+xz)\cancel{(x-z)}$$

$$l = \frac{(x+z)y^2}{y^2+xz}$$

$S = x+z$

$p = xz$

$$l = \frac{sy^2}{y^2+p}$$

$$k = \frac{y+\bar{y}}{2} = \frac{y+\frac{1}{y}}{2} = \frac{y^2+1}{2y}$$

$$\begin{cases} \bar{m} = -m \\ k, l, m \text{ allineati} \end{cases} \quad ②$$

$$\frac{m-l}{m-k} = \frac{\bar{m}-\bar{l}}{\bar{m}-\bar{k}} = \frac{+m+\bar{l}}{+m+\bar{k}}$$

$$(m-l)(m+\bar{k}) = (m+\bar{l})(m-k)$$

$$\cancel{m^2} + \bar{k}m - lm - l\bar{k} = \cancel{m^2} - km + \bar{l}m - k\bar{l}$$

$$k = \bar{k} \qquad m(\bar{l}+l-2k) = k(\bar{l}-l)$$

$$\boxed{m = \frac{k(\bar{l}-l)}{\bar{l}+l-2k}}$$

$$l = \frac{sy^2}{y^2+p}$$

$$\bar{s} = \overline{x+z} = \bar{x}+\bar{z} = \frac{1}{x}+\frac{1}{z} = \frac{z+x}{xz} = \frac{s}{p}$$

$$\bar{p} = \overline{xz} = \frac{1}{zx} = \frac{1}{p}$$

$$\bar{l} = \frac{\bar{s}\,\bar{y}^2}{\bar{y}^2+\bar{p}} = \frac{\frac{s}{p}\cdot\frac{1}{y^2}}{\frac{1}{y^2}+\frac{1}{p}} = \frac{s}{y^2+p}$$

$$\bar{l}-l = \frac{s}{y^2+p}(1-y^2) \qquad \bar{l}+l = \frac{s}{y^2+p}(1+y^2)$$

$$m = \underbrace{\frac{\cancel{y^2+1}}{2y}}_{k} \cdot \underbrace{\frac{s}{y^2+p}(1-y^2)}_{\bar{l}-l} \cdot \frac{1}{\underbrace{\frac{s}{y^2+p}(\cancel{1+y^2})}_{} - \underbrace{\frac{\cancel{y^2+1}}{y}}_{2k}}$$

$$= \frac{s(1-y^2)}{2y\cancel{(y^2+p)}} \cdot \frac{y\cancel{(y^2+p)}}{sy-y^2-p} = \boxed{\frac{s}{2}\,\frac{(1-y^2)}{sy-y^2-p}}$$

$\ell+\ell$

Th $\div$ XKOZ ciclico $\Rightarrow$ YOMI ciclico

ABCD ciclico in complessi:

$$\boxed{\frac{a-b}{c-b}\cdot\frac{c-d}{a-d)}} \in \mathbb{R}$$

$$\frac{x-k}{0-k}\cdot\frac{0-z}{x-z} \in \mathbb{R}$$

$$\frac{x-k}{\cancel{k}}\cdot\frac{z}{x-z} = \frac{\bar{x}-\bar{k}}{\cancel{\bar{k}}}\cdot\frac{\bar{z}}{\bar{x}-\bar{z}}$$

$$z(x-k)\left(\frac{1}{x}-\frac{1}{z}\right) = \left(\frac{1}{x}-k\right)\frac{1}{z}(x-z) \qquad \cdot xz$$

$$-z(x-k)\cancel{(z-x)} = (1-xk)\cancel{(x-z)}$$

$$-zx+zk = 1-xk$$

$$(x+z)k = 1+xz \qquad sk = 1+p \qquad k = \frac{y^2+1}{2y}$$

$$s(y^2+1) = 2y(1+p)$$

$$\boxed{sy^2-2(1+p)y+s=0} \quad =: q(y)$$

YOMI ciclico

$$\frac{y-0}{m-0}\cdot\frac{m-i}{y-i} \in \mathbb{R} \Leftrightarrow \frac{y}{\cancel{m}}\,\frac{(m-i)}{y-i} = \frac{\frac{1}{y}}{\cancel{+m}}\,\frac{(+m+\bar{i})}{\frac{1}{y}-\bar{i}}$$

$$\left(\frac{1}{y}-\frac{1}{i}\right) y\,(m-i) = \frac{1}{y}\left(m+\frac{1}{i}\right)(y-i) \qquad i = x+z$$

$$\left(\frac{1}{y}-\frac{1}{x}-\frac{1}{z}\right) y\,(m-x-z) = \frac{1}{y}\left(m+\frac{1}{x}+\frac{1}{z}\right)(y-x-z)$$

$$y(xz-yx-yz)(m-x-z) = (y-x-z)(mxz+x+z)$$

$$y(p-ys)(m-s) = (y-s)(mp+s)$$

$$m = \frac{1}{2}\,\frac{s(1+y^2)}{sy-p-y^2}$$

$$m-s = \boxed{\frac{1}{2}\,\frac{s}{sy-p-y^2}}\left(1-\cancel{y^2}+2sy+2p+\cancel{2}y^2\right)$$

$$mp+s = \boxed{\frac{1}{2}\,\frac{s}{sy-p-y^2}}\left(p-py^2+2sy-2p-2y^2\right)$$

$$\cancel{y}(p-\cancel{y}s)(1+2sy+2p+\cancel{y^2}) = (y-s)(p-py^2+2sy-2p-2y^2)$$

RHS - RHS = $p(y)$

Hope: $p(y)$ multiplo di $q(y)$

$p(y)$: $(-s)y^4 + (2s^2+2p+2)y^3 + (-5sp-5s)y^2 + (2p^2+2s^2+2p)y - ps$

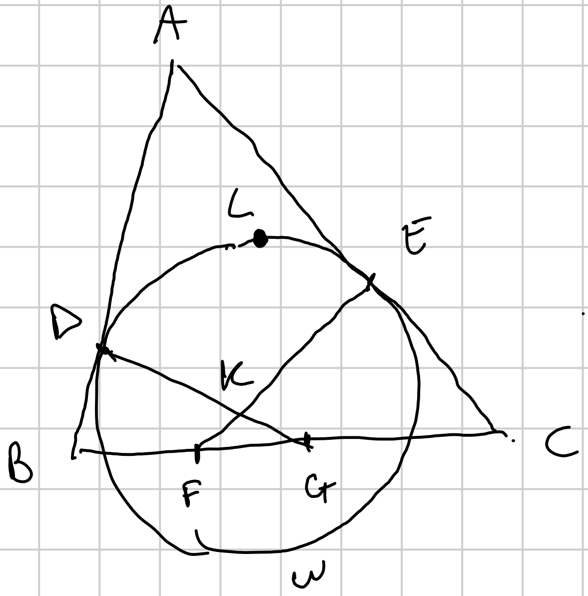
$q(y)$: $sy^2 + (-2(1+p))y + s$

$$\frac{p(y)}{q(y)} = \left(-y^2 + ky - p\right)$$

$$(-y^2+ky-p)(sy^2-2(1+p)y+s) =$$

$$= -sy^4 + sky^3 - psy^2 + 2(1+p)y^3 - 2k(1-p)y^2 + 2p(1+p)y - sy^2 + sky - ps \qquad k = 2s$$

$\square$



$BD = BF$
$CE = CG$

Chiamiamo $t = AD$

Baricentriche su ΔABC

- D, E, F, G facili, anche K
- L? Idea: omotetia che manda A-excerchio in ω

L è l'immagine del punto di tangenza dell' A-excerchio con BC, A_1

$A_1 = [0, s-b, s-c]$ $\quad s = \frac{a+b+c}{2}$

$D = [c-t, t, 0]$ $\qquad E = [b-t, 0, t]$

$F = [0, a-(c-t), c-t]$

$G = [0, b-t, a-(b-t)]$

$EF: (a-b+t)tx - (a-b+t)(c-t)y + (b-t)(c-t)z = 0$

$DG: (a-c+t)tx + (b-t)(c-t)y - (a-c+t)(b-t)z = 0$

FATTO

$$\begin{cases} lx + my + nz = 0 \\ ux + vy + wz = 0 \end{cases}$$ ha soluzione

$[\det X, -\det Y, \det Z]$ dove

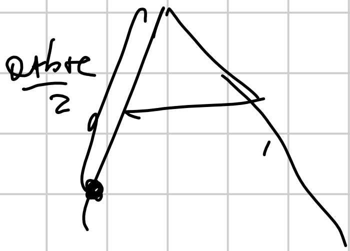
$$X = \begin{pmatrix} m & n \\ v & w \end{pmatrix} \quad Y = \begin{pmatrix} l & n \\ u & w \end{pmatrix}$$

$$Z = \begin{pmatrix} l & m \\ u & v \end{pmatrix}$$

Usando questo,

$$K = \Big[\overset{\alpha}{\underline{(a^2 + 2aT - ab - ac)}}(b-t)(c-t),\; t(b-t)\overset{\beta}{(a+c-b)}(a-c+t),$$
$$t(c-t)\underset{\gamma}{(a-c+b)}(a-b+t) \Big]$$

Ci resta da calcolare L.



Omotetia centro A, fattore t/s

$$\vec{L} = \frac{t}{s}\left(\vec{A_1} - \vec{A}\right) + \vec{A}$$

Attenzione le coordinate di A_1 e A devono avere forma 1

$$A_1 = \left[0, \frac{s-b}{a}, \frac{s-c}{a}\right] \qquad L = [l_1, l_2, l_3]$$

Usando $\ast$,

$$l_1 = \frac{t}{s}(0-1)+1$$

$$l_2 = \frac{t}{s}\left(\frac{s-b}{a}-0\right)+0$$

$$l_3 = \frac{t}{s}\left(\frac{s-c}{a}-0\right)+0$$

$$\Rightarrow \boxed{L = [a(s-t),\ t(s-b),\ t(s-c)]}$$

M

$$\det\begin{pmatrix} a & b & c \\ a(s-t) & t(s-b) & t(s-c) \\ \alpha & \beta & \gamma \end{pmatrix} \begin{array}{l} \to 1 \text{ (grado } t) \\ \to 3 \text{ (grado } t), \end{array}$$

In realtà coeff di t^4 è 0, quindi il determinante è un polinomio di deg 3 in t.

Butto dentro 4 valori di t e se in tutti i casi ottengo 0, allora ho vinto.

Quali possono essere? $t=0$, $t=b$, $t=c$, $t=s-a$

Se $t=0$, $t=b$, $t=c$ così, ok

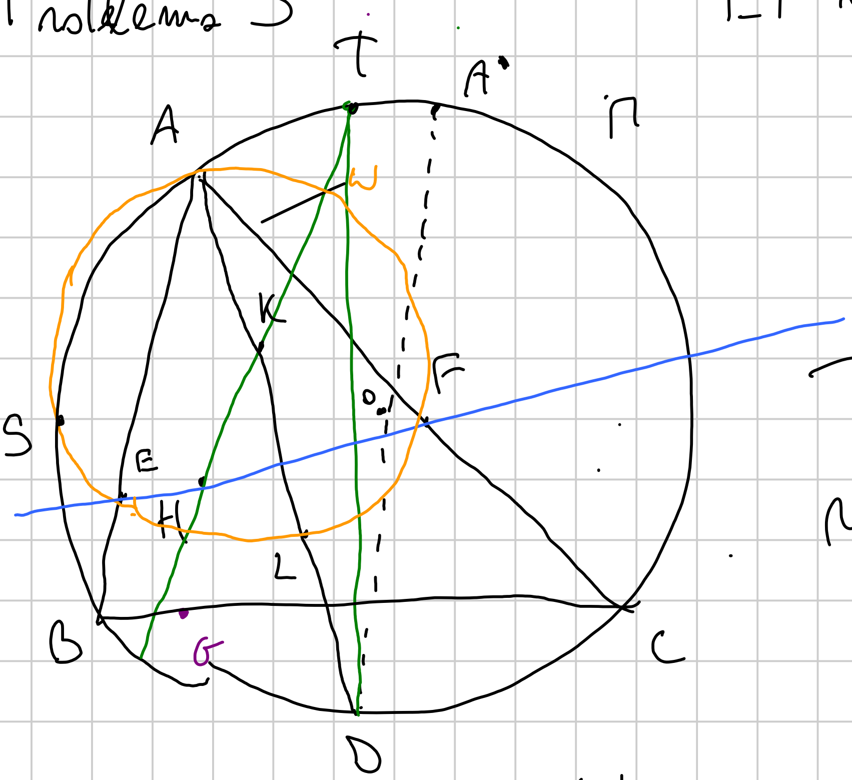
Nell'altro caso

2^a riga $M = 2(1-a)(1-b)(1-c)$ (2^a riga M')

$$\det(M) = 2(1-a)(1-b)(1-c)\det(M')$$

quindi i conti si semplificano.

Problema 3



EF retta che passa per H

K è centro di ω

D: $AK \cap \Gamma$

Tesi: HK, Γ e retta per $D \perp BC$ concorrono

$S = \omega \cap \Gamma$, $\exists$ omotetia in S: $E \to F$, $B \to C$, $\omega \to \Gamma$

$L = AK \cap \omega$, omotetia: $EL \to BD$

ha centro in S: è la stessa

Anche $L \to D$

$A = LK \cap \omega \to DO \cap \Gamma = A'$

S manda $H \to G$, $H \in EF \Rightarrow G \in BC$

S: $A \to A'$, $K \to O$, $H \to G$

$\Delta AKH \simeq \Delta A'OG$

TeP: $DT \perp BC$ (Tesi: T, H, K allineati)

$K' = AD \cap HT$

Rotomotetia in S: $AK'H \simeq A'OG'$

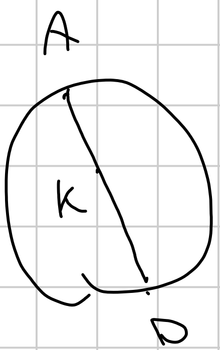
il Hope

Ora se dimostro che $G' = G \Rightarrow A'OG \simeq AKH$

$\Rightarrow K = K' \Rightarrow T, H, K$ allineati.

So che $K' \in$ retta AD, ci basta che $\boxed{G' \in BC}$ NUOVA Tesi

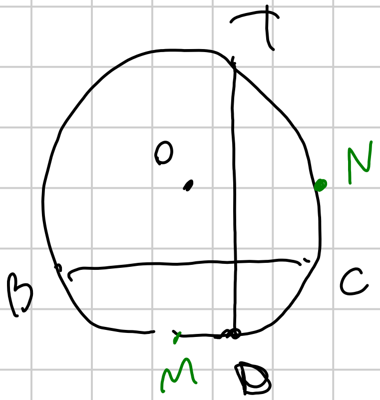
$\Gamma =$ circonferenza unitaria, a, b, c, d



$$K \in AD \iff \frac{k-a}{d-a} = \frac{\bar{k}-\bar{a}}{\bar{d}-\bar{a}} = \frac{\bar{k} - \frac{1}{a}}{\frac{1}{d} - \frac{1}{a}}$$

$$\rightsquigarrow \boxed{k = a + d - ad\bar{k}} \quad (1)$$

$A' \to -d$ Com'è T?



N, M pt medi di BC, TD

$\Rightarrow bc = m^2$, $td = n^2$

$(e^{i\alpha} \cdot e^{i\beta} \to e^{i(\alpha+\beta)})$

$BC \perp TD \iff OM \perp ON$ $\quad n = im$

$td = -bc$. $\quad \boxed{t = -\frac{bc}{d}} \quad \bar{t} = -\frac{d}{bc}$

$H: \to a+b+c$ (perché baricentro è $\frac{a+b+c}{3}$)

$\bar{h} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \bar{a} + \bar{b} + \bar{c}$

$K = AD \cap HT$

$$\frac{K-T}{H-T} = \frac{\bar{k}-\bar{T}}{\bar{H}-\bar{T}} : \quad \frac{k+\frac{bc}{d}}{a+b+c+\frac{bc}{d}} = \frac{\bar{k}+\frac{d}{bc}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}$$

$$K \in AD \rightarrow \bar{k} = \frac{1}{a}+\frac{1}{d}-\frac{k}{ad}$$

Posso scegliere le coordinate in modo che $d = 1$

$$\frac{k+bc}{a+b+c+bc} = \frac{\frac{1}{a}+1-\frac{k}{a}+\frac{1}{bc}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{bc}} = \frac{bc+abc-kbc+a}{ab+bc+ac+a}$$

$$k\Big(\overbrace{\boxed{abc+ac+ab+a}}^{a(b+1)(c+1)} + \overbrace{\boxed{bc+b^2c+bc^2+b^2c^2}}^{bc(b+1)(c+1)}\Big) =$$

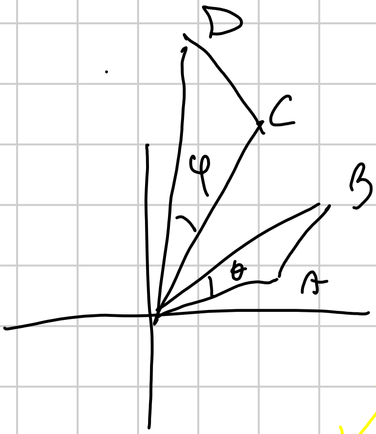
$$= \Big(\underbrace{a^2bc + ab^2c^2}_{abc(a+bc)} + \underbrace{b^2c+bc^2}_{bc(a+b+c)} + \underbrace{a^2+ab+ac+abc}_{a(a+b+c)}\Big)$$

$$(a+bc)(a+b+c)$$

$$k\,(b+1)(c+1)\,(\cancel{a+bc}) = (\cancel{a+bc})\,(a+b+c+abc)$$

$$k = \frac{abc+a+b+c}{(b+1)(c+1)}$$

$AKH \sim A'OG$



$$\frac{b}{a} = \frac{|OB|}{|OA|} \cdot e^{i\theta}$$

$$\frac{d}{c} = \frac{|OD|}{|OC|} \cdot e^{i\varphi}$$

$\Delta OAB \sim \Delta OCD$

$$\frac{k-a}{h-a} = \frac{0-a'}{g-a'} = \frac{1}{g+1}$$

$A' = (-d) = -1 \qquad g+1 = \dfrac{b+c}{k-a}$

$$k - a = \frac{a\cancel{bc} + \cancel{a} + b + c - ab - ac - \cancel{abc} - \cancel{a}}{(b+1)(c+1)} = \frac{b + c - ab - ac}{(b+1)(c+1)}$$

$$= \frac{(b+c)(1-a)}{(b+1)(c+1)}$$

$$g = \frac{(b+1)(c+1)}{1-a} - 1 \qquad \bar{g} = \frac{(\bar{b}+1)(\bar{c}+1)}{1-\bar{a}} - 1$$

$$= \frac{(b+1)(c+1)\,a}{bc(a-1)} - 1$$

GGBC:

$$-\frac{a}{a-1} = -\left(1 + \frac{1}{a-1}\right)$$

$$= -1 + \frac{1}{1-a}$$

$$\Leftrightarrow g = b + c - bc\,\bar{g} =$$

$$\cancel{\frac{(b+1)(c+1)}{1-a}} - 1 = b + c - \frac{(b+1)(c+1)\,a}{a-1} + bc$$

$$= b + c - \frac{(b+1)(c+1)}{1} + \cancel{\frac{(b+1)(c+1)}{1-a}} + bc$$

$$-1 = b + c + bc - (b+1)(c+1)$$ ⇝ che è vero :)