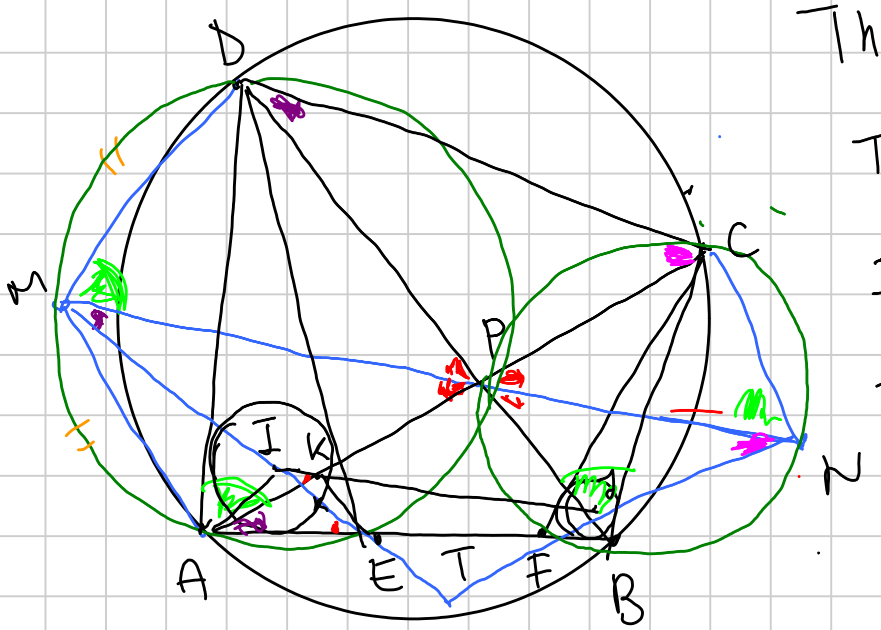
WC 2020 - GEOMETRIA SINTETICA

Note Title 23/01/2020



Th: AEKI ciclico

Th $\Leftrightarrow$ $I\hat{K}A = I\hat{E}A$

$I\hat{E}A = \frac{D\hat{E}A}{2} = \frac{D\hat{P}A}{2}$

Th $\Leftrightarrow$ $MP \parallel IK$

M := bisettrice a $D\hat{P}A$ con (DPEA)

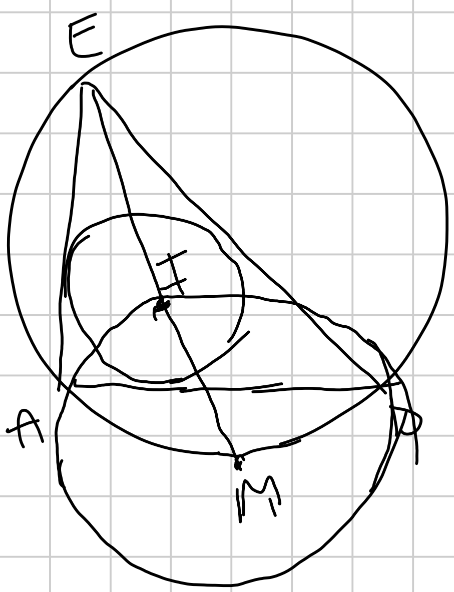
$\overset{\frown}{DM} = \overset{\frown}{MA}$ $\Rightarrow$ E, I, M sono allineati
N, J, F sono allineati

T := $NF \cap ME$

Guardo $\overset{\triangle}{TNM}$ $MN \parallel IJ \Leftrightarrow \frac{TM}{TN} = \frac{IM}{IN}$

$T\hat{M}P = E\hat{A}P = B\hat{D}C$

$\Rightarrow \overset{\triangle}{TMN} \sim \overset{\triangle}{PDC} \Rightarrow \frac{TM}{TN} = \frac{PD}{PC}$



$I\hat{A}D = \frac{1}{2} E\hat{A}D = \frac{1}{2} I\hat{M}D$

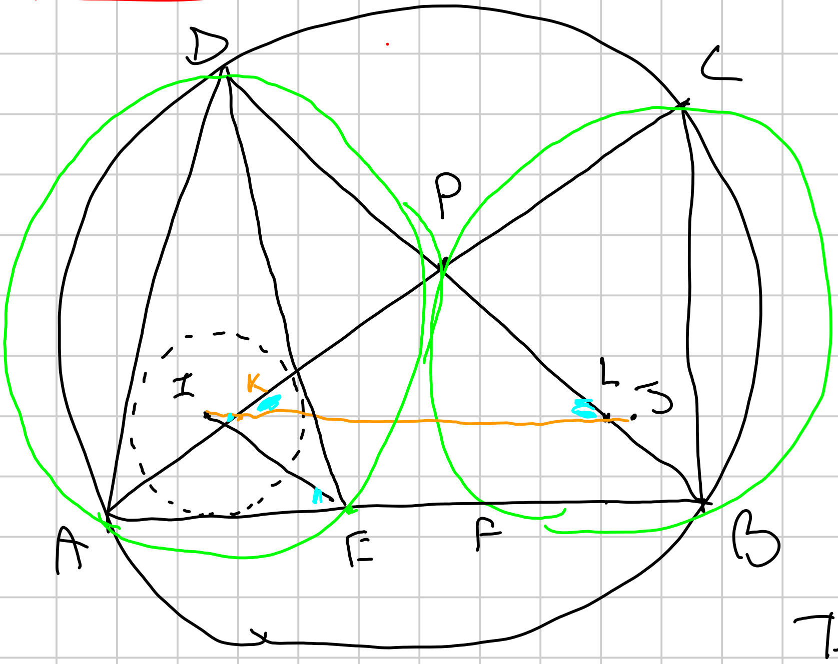
Resta da dimostrare che

$\frac{PD}{PC} = \frac{DM}{CN}$ dato che $IM = DM$
$JN = CN$

Basta dimostrare che

$\triangle DMP \sim \triangle PCN$ che $\hat{DPM} = \hat{CPN} =$ $\hat{DMP} = \hat{CNP} =$

Sol 2



Tesi IKEA sia ciclico

$\angle IKA \overset{?}{=} \angle IEA$

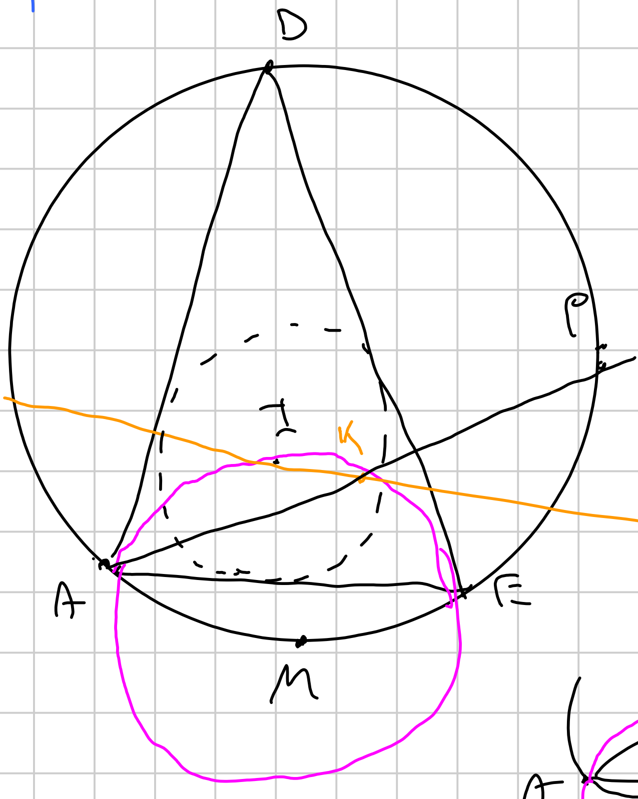
$= \frac{1}{2} \angle AED = \frac{1}{2} \angle APD$

$= \frac{1}{2} \angle CFB \overset{?}{=} \angle JLB$

Tesi vera $\Rightarrow$ PKL è isoscele

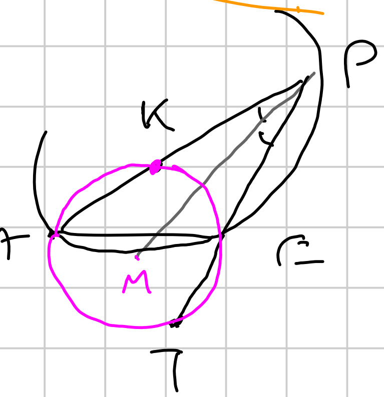
Angle chasing $\Rightarrow \angle PEF = \angle PFE \Rightarrow PE = PF$

Hope: EFKL ciclico di centro P



Hope: $PK = PE$

Fatto: $\bigcirc_{IAE}$ ha centro in M pt medio $\overset{\frown}{AE}$



PM è bisettrice

$PA \to PE$

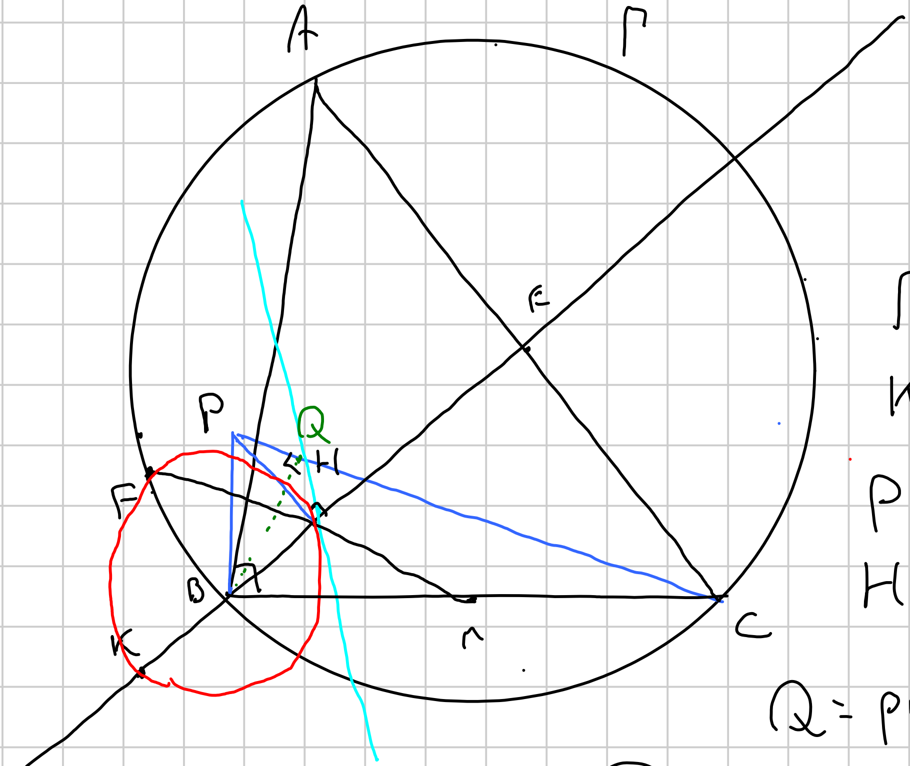
$\bigcirc_{AIE} \to \bigcirc_{AIE}$

riflessione in PM

$\hookrightarrow PE = PK$

$PA = PT$

Problema 2



H è ortocentro di $\triangle ABC$

Γ circoscritto

$F \in \Gamma$: $\angle AFH = 90^\circ$

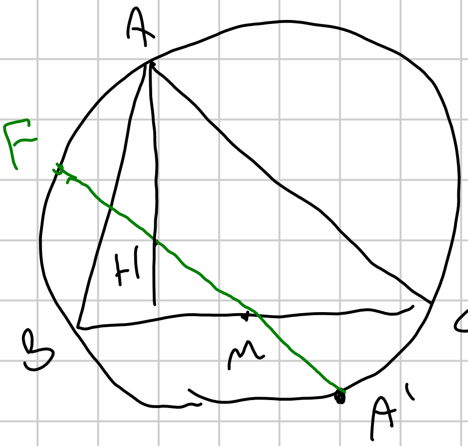
$K \in BH$: $KB = BH$

$PB \perp BC$
$HP \perp BH$ > P è lì

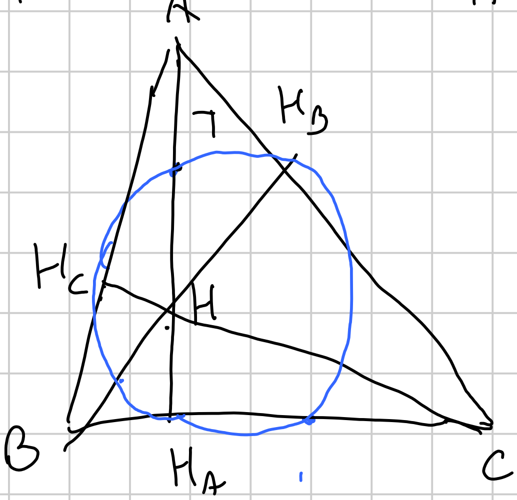
Q = piede da B su PC

Tesi: QH tange $\bigcirc_{FHK}$

Fatto da casa: H, M, A', F allineati



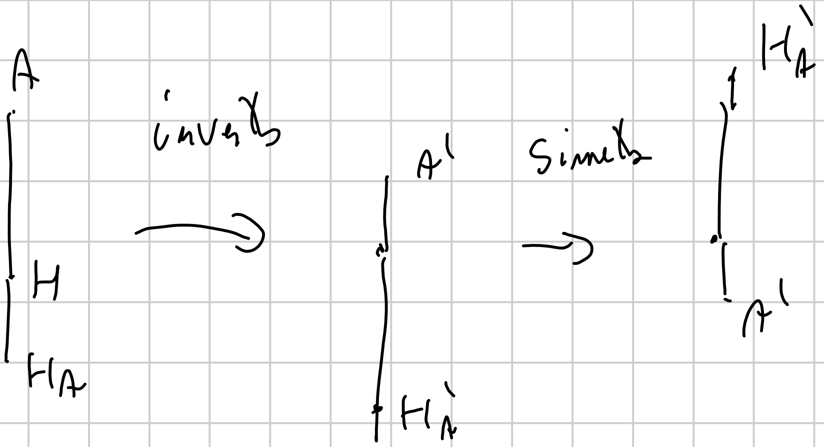
2°: Fatto



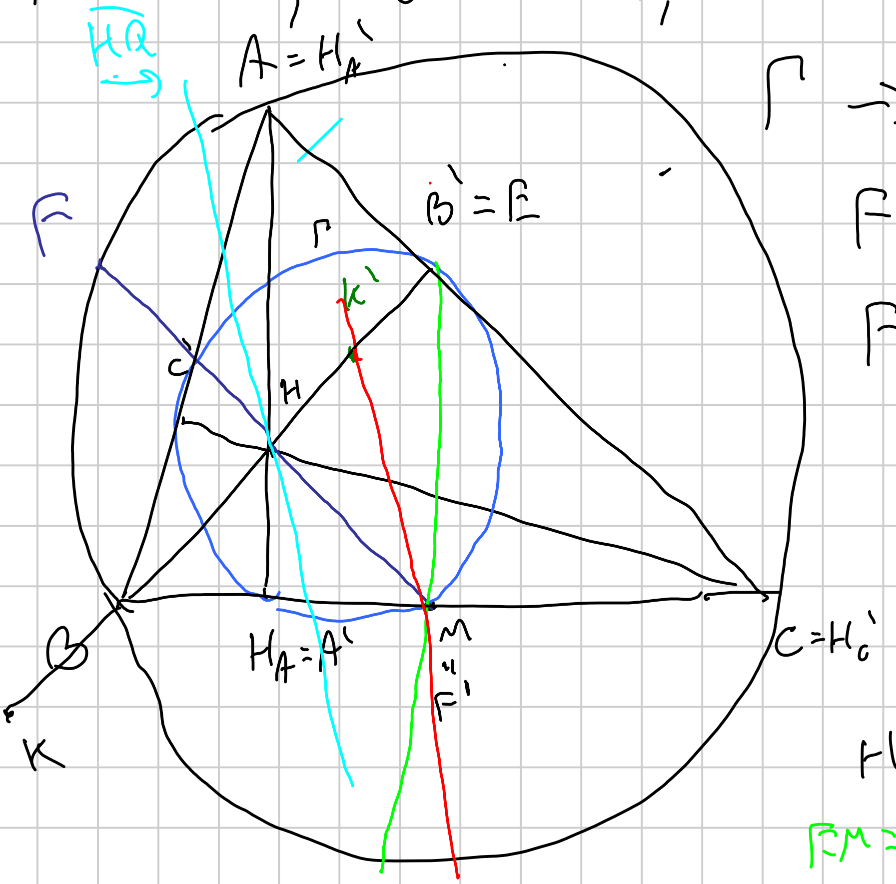
$AH \cdot HH_A = BH \cdot HH_B = CH \cdot HH_C$

$pow_H(\text{Feuerbach}) = HH_A \cdot HT = \frac{1}{2} HH_A \cdot AH$

Facciamo inversione in H di raggio $\sqrt{AH \cdot HH_A}$
+ simmetria centrale:



$A \leftrightarrow H_A$, $B \leftrightarrow H_B$, $C \leftrightarrow H_C$



$\Gamma \to$ Feuerbach.

$F = MH \cap \Gamma$

$F' = M'H \cap \Gamma' = M$

$\to F' \to M$

$HK = 2HB$

$HK' = \frac{1}{2} HB' = \frac{1}{2} HE$

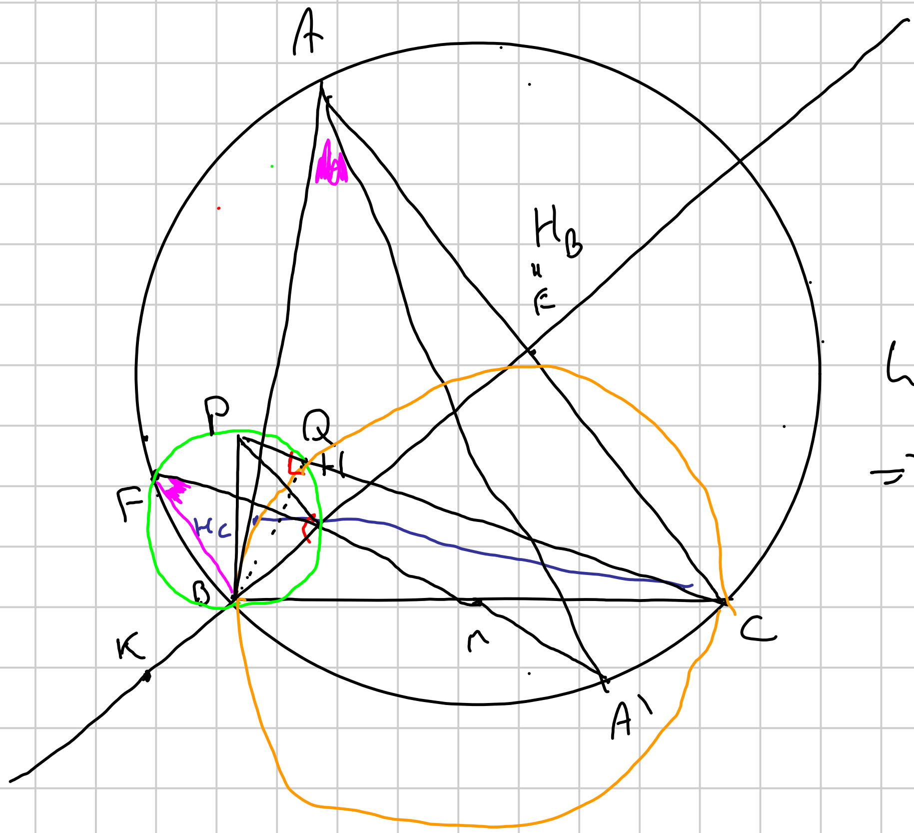
$EM = \gamma'$

$\odot_{FHK} \to$ Retta per $F', M' = KM$

Tesi: HQ tange $\odot_{FHK} \to HQ \parallel KM$

$BQ \perp QC \Rightarrow BQC, H_BH_C$ ciclico

$BQCE$ di centro M. $\to \omega$



Chi è ω'?

$BCE\,H_C$

$\downarrow$

$H_B\,H_C\,B,C$

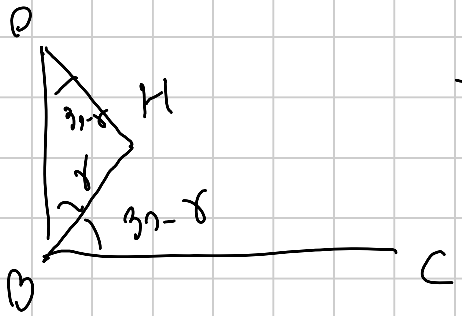
$\omega \to \omega'$

$\Rightarrow Q' \in \omega$

$\angle PQB = \angle PHB = 90^\circ$ Fatto: $PQHBF$ ciclico

• $PQHB$ ciclico $\uparrow$

$\angle BFH = \angle BFA' = \angle BAA' = 90 - \angle AA'B = 90 - \gamma$



$\rightarrow \angle BPH = 90 - \gamma$

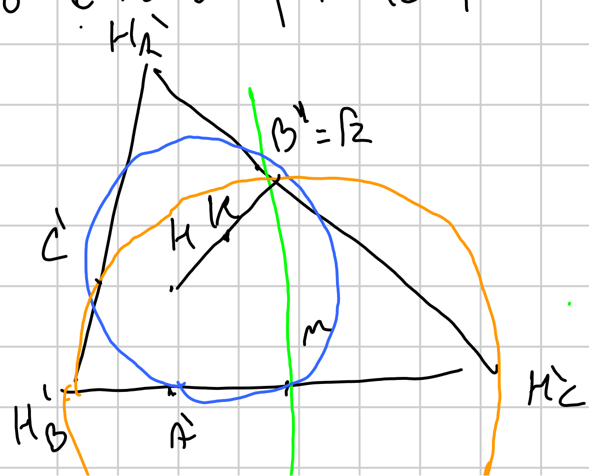
$\downarrow$

$BFPH$ ciclico

$\rightarrow$ la chiamo γ

$Q = \omega \cap \bigcirc_{BFPHQ}$

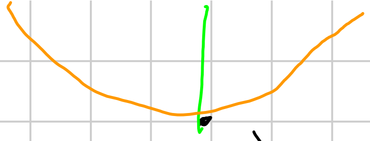
γ' è retta per $B'P' = EM$



$Q' = EM \cap \omega$

Q' è l'opposto di E in ω

$MQ = ME$



Q'

K ~~no~~ pt medio di HE

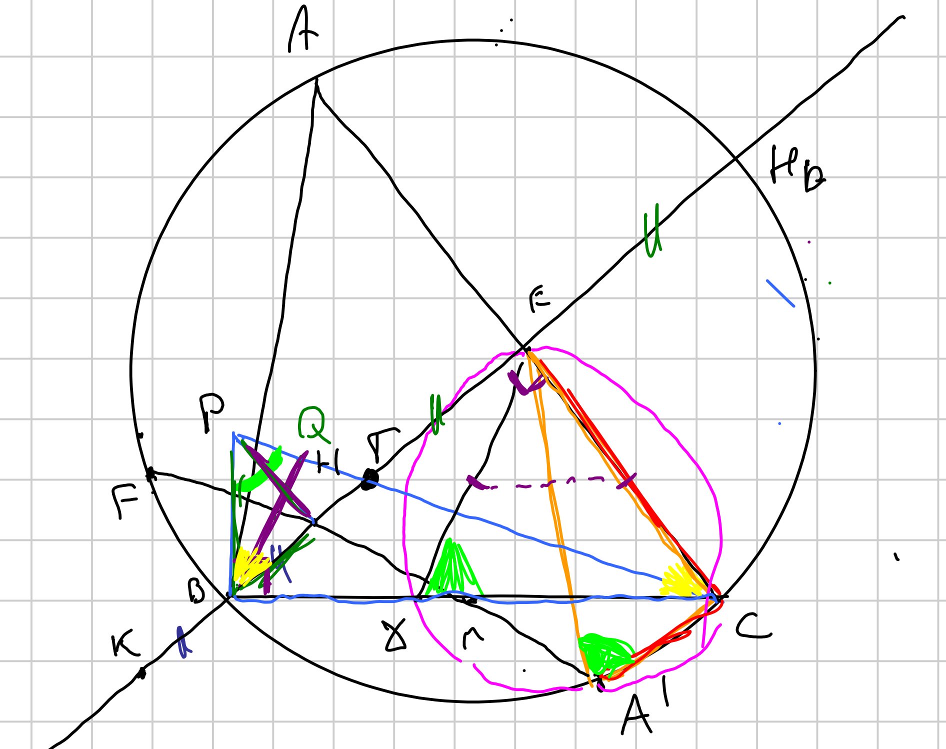
K pt medio EH

Q pt medio EQ

↳ Talete

Tesi $HQ' \parallel KM$

Strada 2 (accenno)



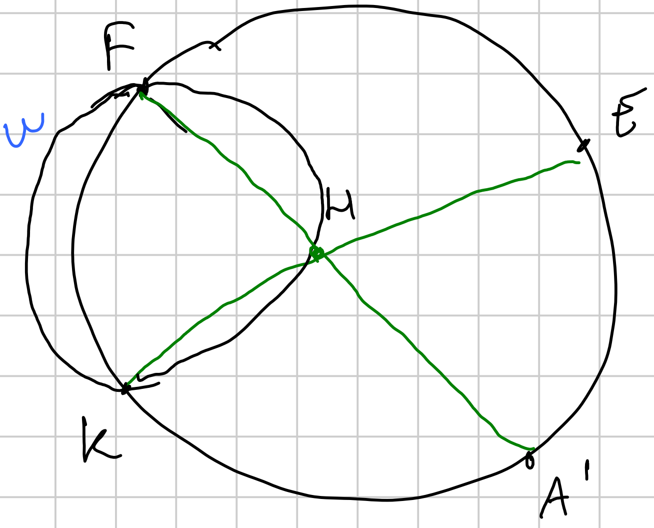
<u>FATTO 1</u> BA'CH è parallelogramma

<u>FATTO 2</u> $HB \cdot HH_b = HB \cdot 2HE = HK \cdot HE$

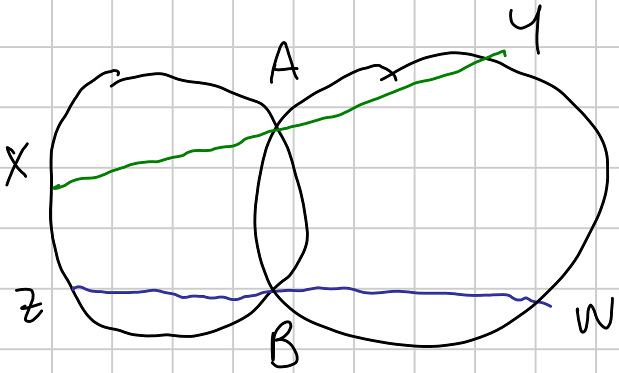
$\parallel$

$FH \cdot HA'$

$\Rightarrow HK \cdot HE = FH \cdot HA' \Rightarrow FKA'E$ ciclico



Teorema (Reim)



Allora $XZ \parallel YW$

Usando Reim, la tesi è equivalente a

$$HQ \parallel EA'$$

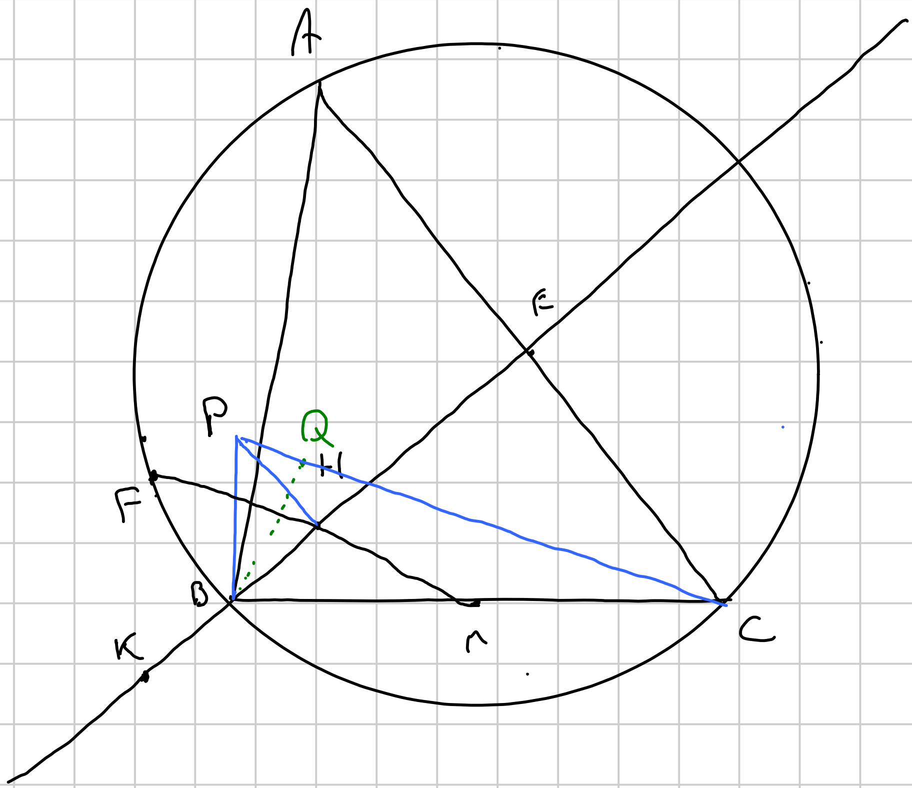
Dimostriamolo.

$$\frac{CA'}{CE} = \frac{HB}{CE} = \frac{BP \cdot \cancel{\cos \angle HBP}}{BC \cdot \cancel{\cos \angle BCA}}$$

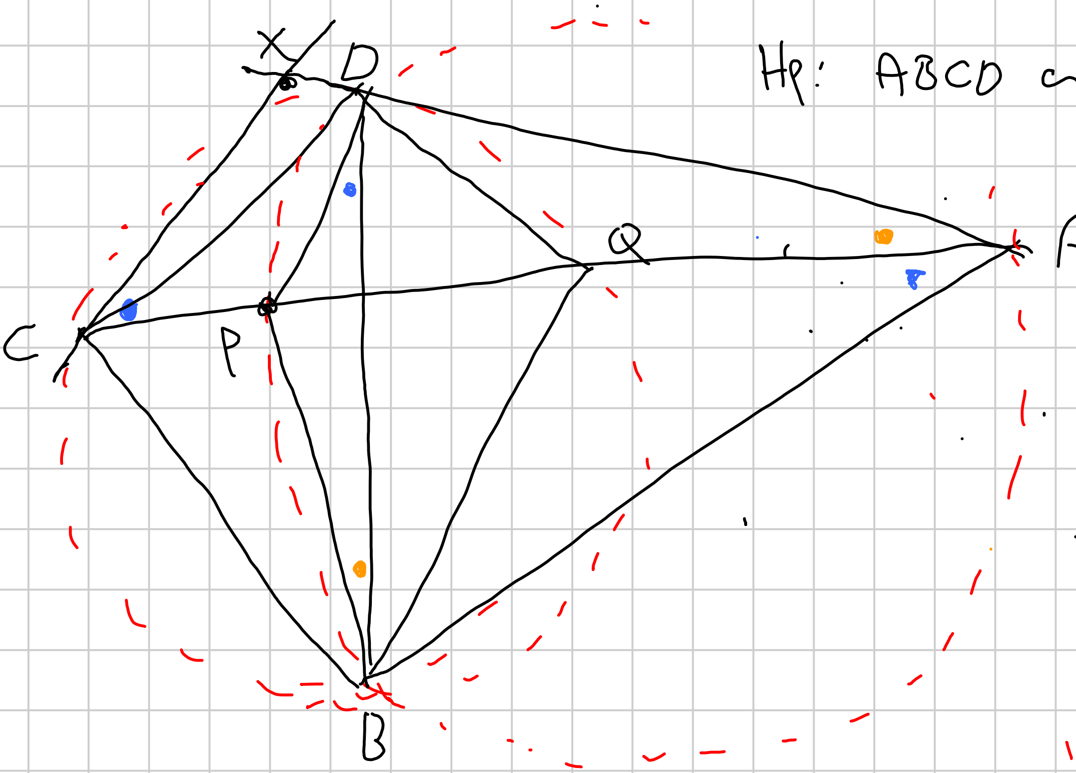
Fatto 1 — guardo Δ rettangoli

$\Rightarrow \quad \Delta BPC \sim \Delta CA'E$

Considerate ΔPBT e ΔCEX (considerate altezze da C, X in ΔCEX)


A
E
P
Q
H
F
B
M
C
K

PROBLEMA 3



Hp: ABCD circoscritto
$\odot(BPDA)$, $\odot(CBQD)$

TS: BPDQ circoscritto.

l = retta per C, $l \parallel AB$, $X = AD \cap l$

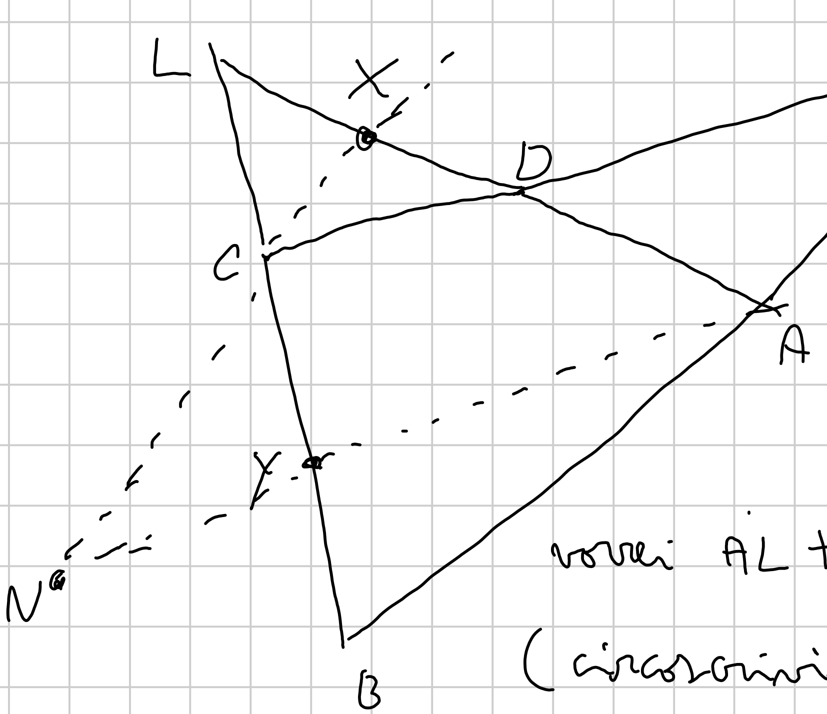
$\Rightarrow X\hat{C}A = C\hat{A}B$. ma $P\hat{B}D = C\hat{A}X$

$\Rightarrow \triangle CAX \simeq \triangle DBP$

Ie l' = retta per A, $l' \parallel CD$. $Y = l' \cap BC$

$\Rightarrow \triangle ACY \simeq \triangle BQD$

$\Rightarrow BPDQ \simeq AXCY$



$L = BC \cap AD$

$M = CD \cap AB$

$N = CX \cap AY$

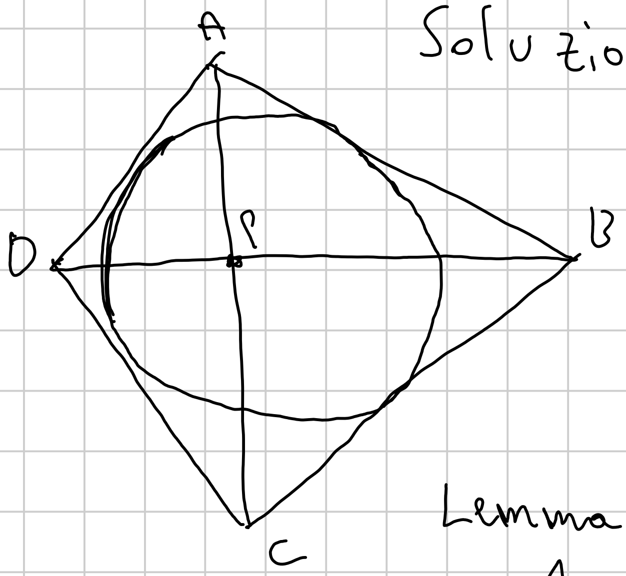
$NAMC$ parallelogramma

vorrei $AL + CN \overset{?}{=} AN + CL$

(circoscrivibilità di $ANCL$)

Ma $AL + CN = AL + AM$ e $AN + CL = CM + CL$

e $ALCM$ è circoscritto $\Rightarrow AL + AM = CL + CM$



Soluzione 2: La tesi è equivalente al seguente fatto: se inverto ABCD di centro P, ottengo un altro quadrilatero circoscritto

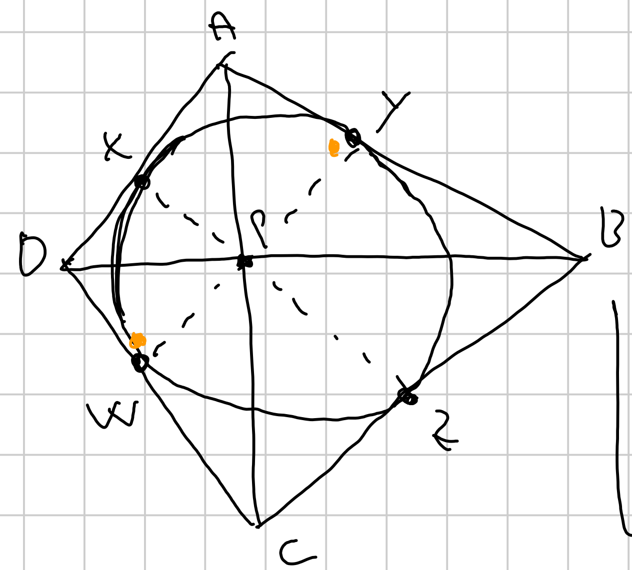
Lemma: ABCD è circoscritto

$$\iff \frac{1}{d(P,AB)} + \frac{1}{d(P,CD)} = \frac{1}{d(P,BC)} + \frac{1}{d(P,AD)}$$

$$PA \cdot PA' = PB \cdot PB' = r^2 \qquad A'B' = \frac{r^2\, AB}{PA \cdot PB}$$

$$h_{AB} \cdot AB = PB \cdot PA \cdot \sin\theta \implies \frac{1}{h_{AB}} = \frac{AB}{PA \cdot PB \cdot \sin\theta} = \frac{A'B'}{r^2 \sin\theta}$$

Quindi Lemma $\Rightarrow$ Tesi



$a = AX = AY$
$b = BY = BZ$
$c = CZ = CW$
$d = DW = DX$

Fatto (noto) X, P, Z
Y, P, W sono allineati

$$\text{Ora} \quad AP/PC = \frac{AP}{a} \cdot \frac{c}{PC} \cdot \frac{a}{c} = \frac{\sin \widehat{AYP}}{\sin \widehat{APY}} \cdot \frac{\sin \widehat{WPC}}{\sin \widehat{CWP}} \cdot \frac{a}{c}$$

$$= \frac{a}{c}$$

$$\implies [APB]/[BPC] = \frac{AP}{PC} = \frac{a}{c}$$

$$\Rightarrow \quad \frac{[APB]}{ab} = \frac{[BPC]}{bc} = \frac{[CPD]}{cd} = \frac{[DPA]}{ad} = \lambda$$

$$\Rightarrow \quad \frac{1}{h_{AB}} = \frac{AB}{2[APB]} = \frac{a+b}{2\lambda ab} = \frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\Rightarrow \quad \frac{1}{h_{AB}} + \frac{1}{h_{CD}} = \frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = \frac{1}{h_{BC}} + \frac{1}{h_{DA}}$$

$\square$