# First day

Mar del Plata, Argentina - July 24, 1997

**1.** In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard).

For any pair of positive integers $m$ and $n$, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths $m$ and $n$, lie along edges of the squares.

Let $S_1$ be the total area of the black part of the triangle and $S_2$ be the total area of the white part. Let

$$f(m,n) = |S_1 - S_2|.$$

(a) Calculate $f(m,n)$ for all positive integers $m$ and $n$ which are either both even or both odd.
(b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all $m$ and $n$.
(c) Show that there is no constant $C$ such that $f(m,n) < C$ for all $m$ and $n$.

**2.** Angle $A$ is the smallest in the triangle $ABC$.

The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$.

The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$.

Show that

$$AU = TB + TC.$$

**3.** Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

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Each problem is worth 7 points.
Time: $4\frac{1}{2}$ hours.