Second day

Mar del Plata, Argentina - July 25, 1997

4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a \geq 1$, $b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents.

Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; $2+2$; $2+1+1$; $1+1+1+1$.

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Each problem is worth 7 points.
Time: $4\frac{1}{2}$ hours.