

**25<sup>th</sup> United States of America Mathematical Olympiad**

**Part I      9 a.m. - 12 noon**

**May 2, 1996**

1. Prove that the average of the numbers  $n \sin n^\circ$  ( $n = 2, 4, 6, \dots, 180$ ) is  $\cot 1^\circ$ .
2. For any nonempty set  $S$  of real numbers, let  $\sigma(S)$  denote the sum of the elements of  $S$ . Given a set  $A$  of  $n$  positive integers, consider the collection of all distinct sums  $\sigma(S)$  as  $S$  ranges over the nonempty subsets of  $A$ . Prove that this collection of sums can be partitioned into  $n$  classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.
3. Let  $ABC$  be a triangle. Prove that there is a line  $\ell$  (in the plane of triangle  $ABC$ ) such that the intersection of the interior of triangle  $ABC$  and the interior of its reflection  $A'B'C'$  in  $\ell$  has area more than  $2/3$  the area of triangle  $ABC$ .

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**Part II      1 p.m. - 4 p.m.**

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4. An  $n$ -term sequence  $(x_1, x_2, \dots, x_n)$  in which each term is either 0 or 1 is called a *binary sequence of length  $n$* . Let  $a_n$  be the number of binary sequences of length  $n$  containing no three consecutive terms equal to 0, 1, 0 in that order. Let  $b_n$  be the number of binary sequences of length  $n$  that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that  $b_{n+1} = 2a_n$  for all positive integers  $n$ .
5. Triangle  $ABC$  has the following property: there is an interior point  $P$  such that  $\angle PAB = 10^\circ$ ,  $\angle PBA = 20^\circ$ ,  $\angle PCA = 30^\circ$ , and  $\angle PAC = 40^\circ$ . Prove that triangle  $ABC$  is isosceles.
6. Determine (with proof) whether there is a subset  $X$  of the integers with the following property: for any integer  $n$  there is exactly one solution of  $a + 2b = n$  with  $a, b \in X$ .