A - Numeri Complessi

Titolo nota 05/11/2018

I numeri complessi : $a+ib$ con $a,b\in\mathbb{R}$ i unità immaginaria t.c. $\boxed{i^2=-1}$ $i=\sqrt{-1}$

$\mathbb{C}=\{a+ib \mid a,b\in\mathbb{R}\}$.

$z\in\mathbb{C}$ $z=a+ib$ $a=\mathrm{Re}(z)$ parte reale
$b=\mathrm{Im}(z)$ parte immaginaria

$z\in\mathbb{C} \rightsquigarrow \bar{z}\in\mathbb{C}$ coniugato

$z=a+ib$
$\bar{z}=a-ib$ $\mathrm{Re}(z)=\mathrm{Re}(\bar{z})$
$\mathrm{Im}(z)=-\mathrm{Im}(\bar{z})$

$$\mathrm{Re}(z)=\frac{z+\bar{z}}{2} \qquad \mathrm{Im}(z)=\frac{z-\bar{z}}{2i}$$

$$\frac{z+\bar{z}}{2}=\frac{(a+ib)+(a-ib)}{2}=\frac{a+\cancel{ib}+a-\cancel{ib}}{2}=\frac{2a}{2}=a$$

$$z\cdot\bar{z}=(a+ib)(a-ib)=a^2-(ib)^2=a^2-i^2b^2 \overset{i^2=-1}{=} a^2+b^2\geq 0$$

$|z|=\sqrt{z\cdot\bar{z}}=\sqrt{a^2+b^2}$ modulo

è uguale a zero se e solo se $a=b=0$ se e solo se $z=0$.

Operazioni tra numeri complessi

$z=a+ib$ $w=c+id$

- $z\pm w=(a+ib)\pm(c+id)=(a\pm c)+i(b\pm d)$
- $z\cdot w=(a+ib)(c+id)=ac+i(ad)+i(bc)+i^2bd=$
  $=(ac-bd)+i(bc+ad)$.
- $\dfrac{z}{w}=\dfrac{a+ib}{c+id}\cdot\dfrac{c-id}{c-id}=\dfrac{(ac+bd)}{c^2+d^2}+i\dfrac{(bc-ad)}{c^2+d^2}$

Es: $(1+i)(2-i)+\dfrac{1-i}{2+3i}=$

$=(2+1)+i(-1+2)+\dfrac{(1-i)(2-3i)}{4+9}=3+i+\dfrac{(2+3)-i(2+3)}{13}=$

$=\left(3+\frac{5}{13}\right)+i\left(1-\frac{5}{13}\right)$

Proprietà delle operazioni

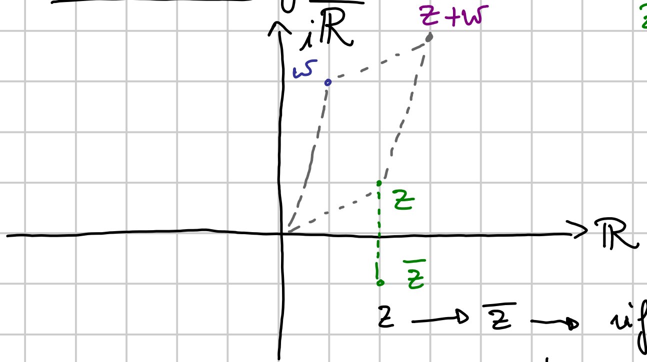
- $\overline{z+w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$
- $|zw| = |z| \cdot |w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $|z+w| \le |z| + |w|$

Oss: $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ idem per Im

$\boxed{\cancel{\mathrm{Re}(zw) = \mathrm{Re}(z)\,\mathrm{Re}(w)}}$

$$\mathrm{Re}(zw) = \mathrm{Re}(z)\,\mathrm{Re}(w) - \mathrm{Im}(z)\,\mathrm{Im}(w)$$

Piano di Gauss



$z = 2+i \to (2,1)$ $\qquad$ $w = 1+3i \to (1,3)$

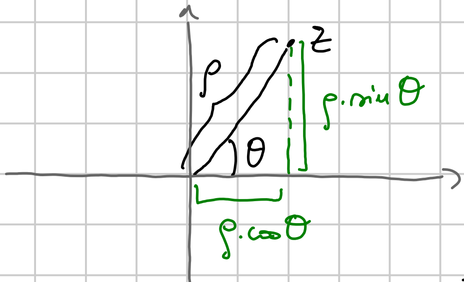
$z+w = 3+4i \to (3,4)$

$z \to \bar{z} \to$ riflessione risp all'asse x

$|z| =$ distanza dall'origine

Forma trigonometrica / polare / esponenziale (vs. forma cartesiana)

↳ $a+ib$



$$z = \rho(\cos\theta + i\sin\theta) = \rho \cdot e^{i\theta}$$

Def: $e^{i\theta} = \cos\theta + i\sin\theta$

θ si dice argomento di z $\quad (\arg(z) = \theta)$

$z = \rho_1(\cos\theta_1 + i\sin\theta_1) \qquad w = \rho_2(\cos\theta_2 + i\sin\theta_2)$

$$zw = \rho_1\rho_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right)$$

$$= \rho_1\rho_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

$$(\rho_1 e^{i\theta_1}) \cdot (\rho_2 e^{i\theta_2}) = \rho_1\rho_2\, e^{i(\theta_1+\theta_2)}$$

$|zw| = |z| \cdot |w| \qquad \arg(zw) = \arg(z) + \arg(w)$ a meno di multipli di 2π

$[\arg(zw) \equiv \arg(z) + \arg(w) \pmod{2\pi}]$

Es: $\dfrac{1+\sqrt{3}i}{2+2i} = \dfrac{2}{2\sqrt{2}}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) = \dfrac{1}{\sqrt{2}}\left(\cos\frac{\pi}{12} + i\,\mathrm{sen}\frac{\pi}{12}\right)$

$$(1+\sqrt{3}i) = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$2+2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4}+i\,\mathrm{sen}\frac{\pi}{4}\right)$$

Om: $\dfrac{1+i\sqrt{3}}{2+2i} = \dfrac{(1+i\sqrt{3})(2-2i)}{8} = \dfrac{(2+2\sqrt{3})+i(2\sqrt{3}-2)}{8} =$

$$= \left(\frac{1+\sqrt{3}}{4}\right) + i\left(\frac{\sqrt{3}-1}{4}\right)$$

$$\frac{1+\sqrt{3}}{4} = \frac{1}{\sqrt{2}}\cos\frac{\pi}{12}$$

$$\frac{\sqrt{3}-1}{4} = \frac{1}{\sqrt{2}}\sin\frac{\pi}{12}$$

Potenze di un numero complesso

$$(a+ib)^n = \ldots\ldots$$

$$i^0 = 1,\ i^1 = 1,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \ldots$$

$$z = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$

$$z^n = \rho^n(\cos(n\theta) + i\sin(n\theta)) = \rho^n e^{in\theta}$$

Fare la radice è più complicato

Oss: Se $|z| = 1$, allora $\dfrac{1}{z} = \bar{z}$

infatti $|z| = 1$ se e solo se $z\bar{z} = 1 \implies \bar{z} = \dfrac{1}{z}$