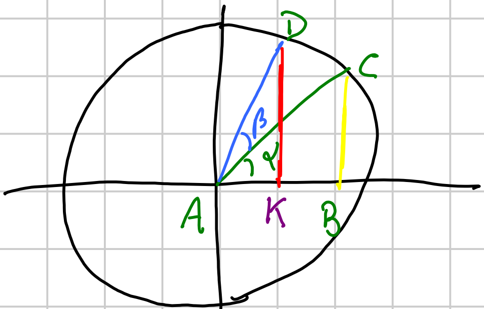
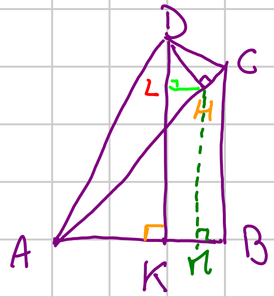
Pillole di Trigonometria 2

Titolo nota 01/12/2018





$\sin(\alpha+\beta) = DK$ $\quad DL \quad L\hat{D}H \quad A\hat{K}D = \frac{\pi}{2} = A\hat{H}D \Rightarrow AKHD$ è ciclico

$L\hat{D}H = K\hat{D}H = K\hat{A}H = B\hat{A}C = \alpha$

Essendo $AD = 1$ e $AC = 1$ e $C\hat{A}D = \beta$ vale $DH = \sin\beta$

$DL = DH \cdot \cos(L\hat{D}H) = \sin\beta \cdot \cos\alpha$

$KL = HM$. Ora i triangoli ABC e AMH sono simili

$\frac{BC}{AC} = \frac{HM}{AH}$ $\quad AC = 1$, invece $AH = \cos\beta$

$HM = BC \cdot AH = \sin\alpha \cos\beta$

Dunque $\sin(\alpha+\beta) = DK = DL + KL = DL + HM = \sin\beta\cos\alpha + \sin\alpha\cos\beta$

$\sin(\alpha-\beta) = \sin(-\beta)\cos\alpha + \sin\alpha\cos(-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

$\cos(\alpha+\beta) = \sin\left(\frac{\pi}{2} - (\alpha+\beta)\right) = \sin\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right) = \sin\left(\frac{\pi}{2}-\alpha\right)\cos\beta - \sin\beta\cos\left(\frac{\pi}{2}-\alpha\right) =$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$tg(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}} = \frac{tg\alpha + tg\beta}{1 - tg\alpha \cdot tg\beta}$$

$$tg(\alpha-\beta) = \frac{tg\alpha + tg(-\beta)}{1 - tg\alpha \cdot tg(-\beta)} = \frac{tg\alpha - tg\beta}{1 + tg\alpha \cdot tg\beta}$$

$$cotg(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha} = \frac{\frac{\cos\alpha}{\sin\alpha}\cdot\frac{\cos\beta}{\sin\beta} - 1}{\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}} =$$

$$= \frac{\cot\alpha \cdot \cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$\cot(\alpha-\beta) = \frac{-\cot\alpha \cot\beta - 1}{\cot\alpha - \cot\beta}$$

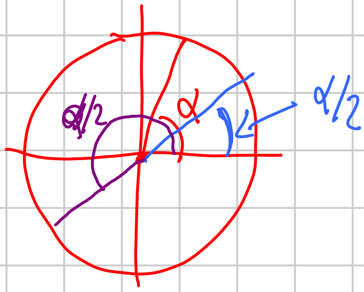
Duplicazione degli angoli:

$$\sin 2\alpha = \sin(\alpha+\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos(\alpha+\alpha) = (\cos\alpha)^2 - (\sin\alpha)^2 = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$tg(2\alpha) = tg(\alpha+\alpha) = \frac{2\,tg\alpha}{1-tg^2\alpha}$$

E se li dimezzassimo?



$2\pi+\alpha$

$\sin(\gamma+\pi) = -\sin\gamma \quad \cos(\gamma+\pi) = -\cos\gamma$

$(\text{"}\gamma = \frac{\alpha}{2}\text{"})$

$$\cos\alpha = \cos\left(2\cdot\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$t = tg\frac{\alpha}{2} \qquad t = tg\frac{\alpha}{2} = \frac{\sin\alpha/2}{\cos\alpha/2} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$t = \frac{\sin\alpha/2}{\cos\alpha/2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$t^2 = \frac{1-\cos\alpha}{1+\cos\alpha} \qquad t^2 + t^2\cos\alpha = 1-\cos\alpha \qquad \cos\alpha = \frac{1-t^2}{1+t^2}$$

$$\sin\alpha = \frac{1-\cos\alpha}{t} = \frac{1-\frac{1-t^2}{1+t^2}}{t} = \frac{\frac{2t^2}{1+t^2}}{t} = \frac{2t}{1+t^2}$$

$$1 \overset{?}{=} \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 = \frac{4t^2+1-2t^2+t^4}{t^4+2t^2+1} = 1$$

Prostaferesi:

- $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
- $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
- $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
- $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

(I) $RHS = 2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) =$

$= 2\left(\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin\frac{\beta}{2}\cos\frac{\beta}{2}\right) = \sin\alpha + \sin\beta = LHS$

Werner

- $\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$
- $\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$
- $\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

(I) $\gamma = \alpha+\beta \quad \delta = \alpha-\beta$ (Prostaferesi con γ e δ)

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\left[\sin\alpha\cos\beta\right]$

$\sin\frac{\pi}{8}$, $\cos\pi/8$

$\alpha = \pi/4 \Rightarrow \frac{\pi}{8} = \alpha/2$

$\sin\frac{\pi}{8}$

$= \pm\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

segno + perchè $0 < \pi/8 < \pi/2$

$\cos\frac{\pi}{8} = \pm\sqrt{\frac{1+\cos\pi/4}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$