

Runge

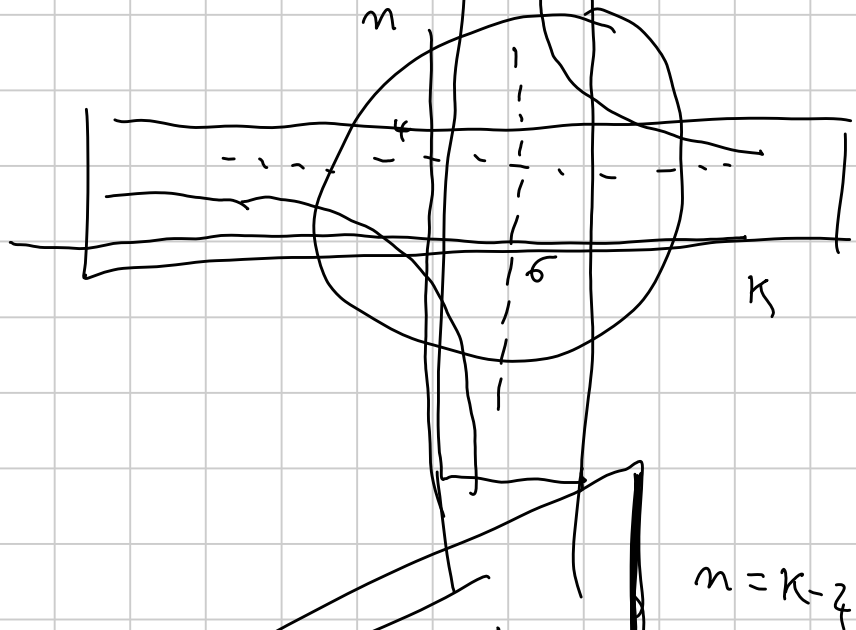
$$6n + 7 = k(n - 4)$$

$$k = \frac{6n + 7}{n - 4} = 6 + \frac{31}{n - 4}$$

$$n - 4 = 31 \quad n = 35$$

$$\frac{k - 6}{n - 4} = \frac{31}{n - 4}$$

$$(k - 6)(n - 4)$$



$$n(k - n) = 4k + 7$$



$$mn \quad n - k + 4 = \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix}$$

Strategia :

Strategia:

$$F(x, y) = 0 \quad F \in \mathbb{Z}[x, y] \text{ irriducibile}$$

Cerchiamo dei polinomi $G_1, \dots, G_n \in \mathbb{Z}[x, y]$ non costanti e non divisibili per F e delle costanti $M_0, \dots, M_n > 0$ reali tali che:

$$\mathcal{C} = \{ (x, y) \in \mathbb{R}^2 \mid F(x, y) = 0 \}$$

$$U_i = \{ (x, y) \in \mathbb{R}^2 \mid |G_i(x, y)| \leq M_i \} \quad i=1, \dots, n$$

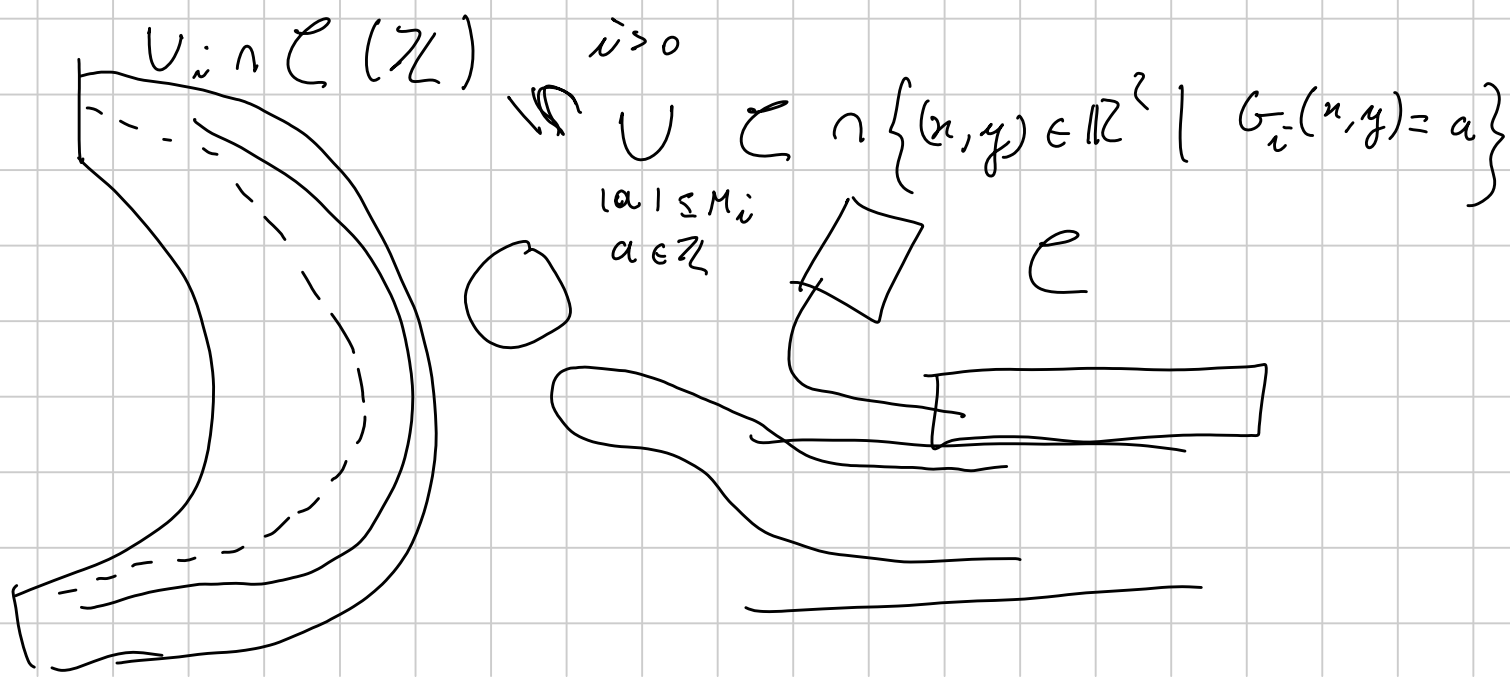
$$U_0 = \{ (x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) \leq M_0 \}$$

$$\mathcal{C} \subseteq \bigcup_{i=0}^n U_i$$

Allora $\mathcal{C}(\mathbb{Z}) := \{ (x, y) \in \mathbb{Z}^2 \mid F(x, y) = 0 \}$

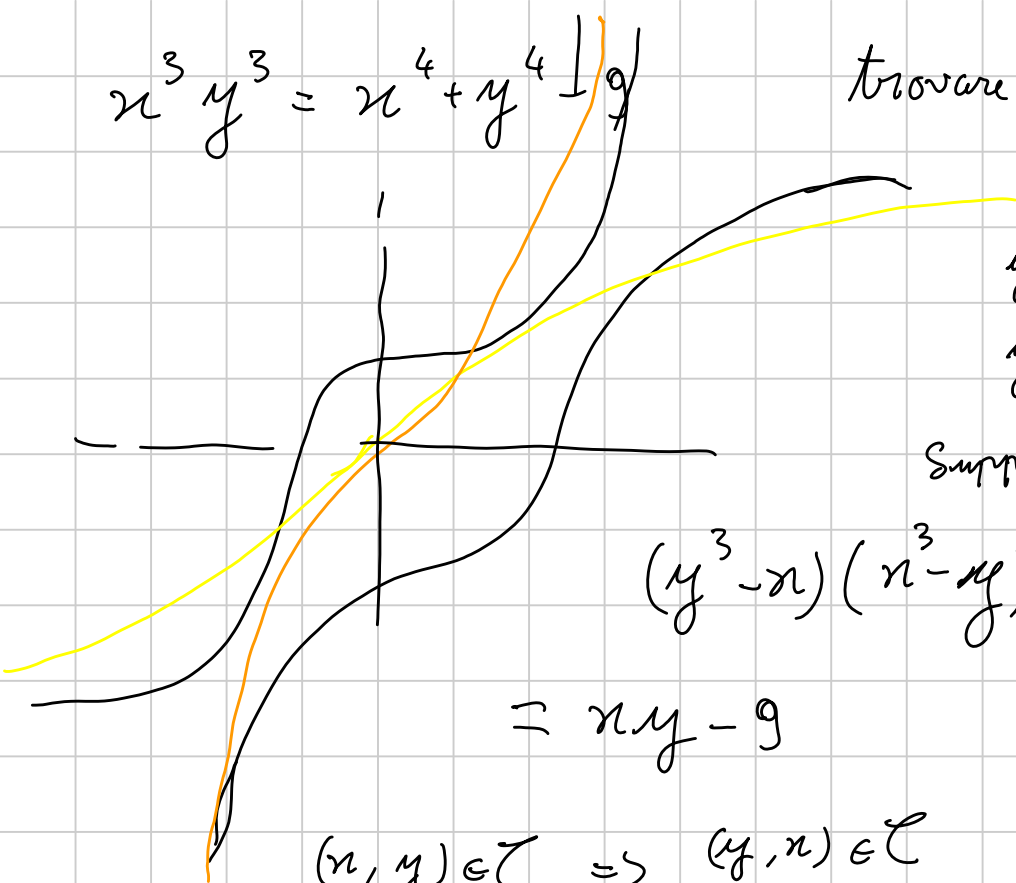
è finito e lo possiamo trovare

$$\mathcal{C}(\mathbb{Z}) = \bigcup_{i=0}^n (U_i \cap \mathcal{C}(\mathbb{Z})) \quad U_0 \cap \mathcal{C}(\mathbb{Z}) \quad \checkmark$$



$$x^3 y^3 = x^4 + y^4 + 9$$

trovare le sol. intere



$$y^3 = x$$

$$y = x^3$$

Supponiamo che $(x, y) \in \mathbb{Z}$

$$(y^3 - x)(x^3 - y) = x^3 y^3 - x^4 - y^4 + xy$$

$$= xy - 9$$

$$(x, y) \in \mathbb{Z} \Rightarrow (y, x) \in \mathbb{Z} \\ (-x, -y) \in \mathbb{Z}$$

Se $|x|, |y| \geq 2$
allora $xy > 0$

assumiamo $3 \leq x \leq y$ vediamo che $x^3 - y$ è piccola

$$0 \leq x^3 - y = \frac{xy - 9}{y^3 - x} < \frac{y^2}{y^3 - y} = \frac{y}{y^2 - 1} \leq \frac{3}{8}$$

$$\text{Se } x^3 = y$$

$$x^2 = x^4 + x^2 - 9$$



Rimangono

$$x=0$$

$$y^4 = 9$$



$$x=1$$

$$y^4 - y^3 - 8 = 0$$

$$y^3 \mid 8$$

$$y = \pm 1, \pm 2$$

$$x \geq 2$$

$$y^2 - 8y^3 + 7 = 0$$

$$y^3 \mid 7$$

$$x=1, y=2$$

$$y = \pm 1$$

$$x=2, y=1$$

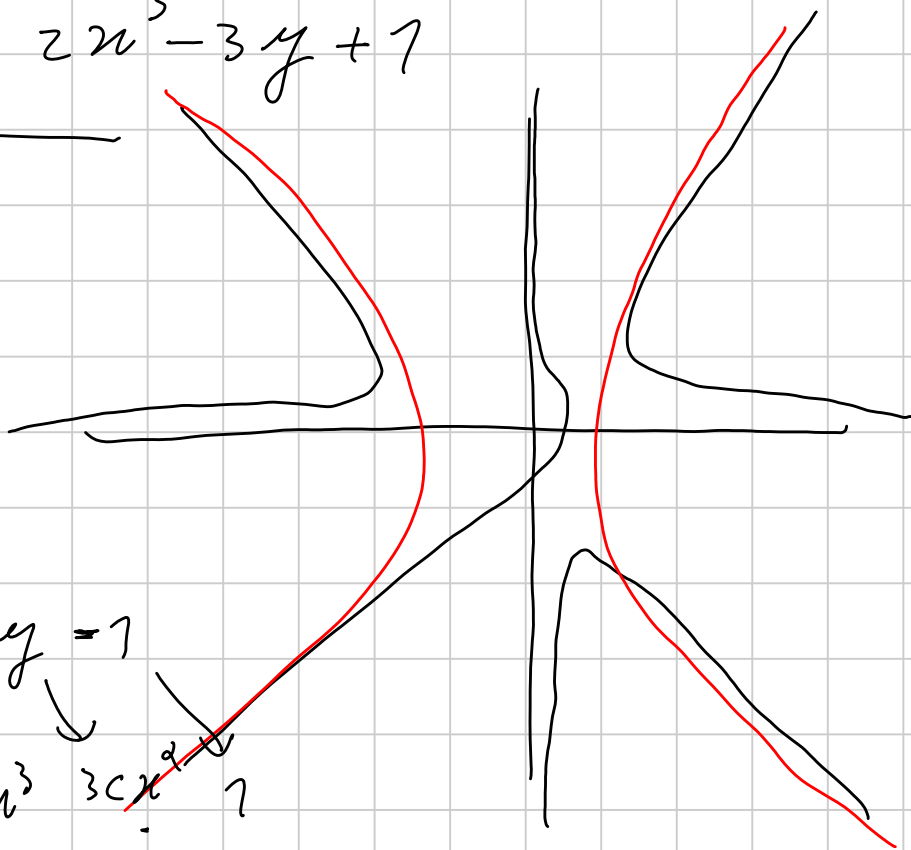
$$(1, 2) \quad (2, 1)$$

$$(-1, -2) \quad (-2, -1)$$

$$(x^2 - 2y^2)xy = 2x^3 - 3y + 1$$

$x \rightarrow \infty$

$$y \sim \frac{cx^\alpha}{a \in \mathbb{Q}} \quad \left| \frac{cx^\alpha}{y} \right| \rightarrow 1$$



$$x^3 y - 2x y^3 - 2x^3 + 3y = 1$$

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ c x^{3+\alpha} & -2c^3 x^{1+3\alpha} & -2x^3 & 3cx^\alpha & 1 \end{array}$$

$\alpha > 0$ See $\alpha > 0$

$$3+\alpha = 1+3\alpha \quad \alpha = 1$$

$$c - 2c^3 = 0 \rightarrow c = \pm \frac{1}{\sqrt{2}}$$

See $\alpha < 0$

$$1+3\alpha = \alpha \quad \alpha = -1/2$$

$$y_1 = \frac{x}{\sqrt{2}} + a + \frac{b}{x} + O(1/x^2) \quad y_1^2 = \frac{x^2}{2} + \frac{2a}{\sqrt{2}}x + \left(a^2 + \frac{2b}{\sqrt{2}}\right) + O(1/x)$$

$$x^2 - 2y_1^2 = -\frac{4a}{\sqrt{2}}x - 2\left(a^2 + \frac{2b}{\sqrt{2}}\right) + O(1/x)$$

$$(x^2 - 2y_1^2)xy_1 = \left(\frac{x}{\sqrt{2}} + a + \frac{b}{x} + O(1/x^2)\right) \left(-\frac{4a}{\sqrt{2}}x^2 - 2\left(a^2 + \frac{2b}{\sqrt{2}}\right)x + O(1)\right)$$

$$= -2ax^3 + \left[-\frac{4a^2}{\sqrt{2}} - \frac{2}{\sqrt{2}}\left(a^2 + \frac{2b}{\sqrt{2}}\right)\right]x^2 + O(x)$$

$$= 2x^3 - 3y + 1 = 2x^3 + O(x)$$

$$-2a = 2 \quad a = -1 \quad -\frac{4^2}{\sqrt{2}} - \frac{1}{\sqrt{2}} \left(1 + \frac{2b}{\sqrt{2}}\right) = 0$$

$$-2 = 1 + \frac{2b}{\sqrt{2}} \quad -\frac{3}{2} \sqrt{2} = b$$

$$y_1 = \frac{x}{\sqrt{2}} - 1 = \frac{3}{\sqrt{2}} \frac{1}{n} + O(1/n^2)$$

$$y_2 = -\frac{x}{\sqrt{2}} - 1 + \frac{3}{\sqrt{2}} \frac{1}{n} + O(1/n^2)$$

$$y_1 + y_2 = -2 + O(1/n^2)$$

$$y_1 \cdot y_2 = -\frac{1}{2} x^2 + \left(1 + \frac{3}{2} + \frac{3}{2}\right) + O(1/n)$$

$$y_2 = -2 - y_1 + O(1/n^2)$$

$$y_2 (2 - y_2 + O(1/n^2)) = -\frac{1}{2} x^2 + 4 + O(1/n)$$

$$-y_1^2 - 2y_1 + \frac{1}{2} x^2 - 4 = O(1/n)$$

$$2y^2 + 4y - x^2 + 8 = O(1/n)$$

$$x^2 - 2y^2 - 4y - 8$$

$$(x^2 - 2y^2 - 4y - 8) (x) (y-2) =$$

$$= \dots = \underline{26x - 3y + 1}$$

$$\text{se } |x|, |y-2| \geq 5$$

$$|x^2 - 2y^2 - 4y - 8| \leq \frac{16}{|y-2|} + \frac{3}{|x|} \left| \frac{y^{-2+2}}{y-2} \right| + \frac{1}{|x||y-2|}$$

$$\leq \frac{16}{5} + \frac{1}{25} + \frac{3}{5} + \frac{6}{25} < 5$$

Vale una Tra

$$\left[\begin{array}{l} |x| \leq 4 \\ |y-2| \leq 4 \\ |x^2 - 2y^2 - 4y - 8| \leq 4 \end{array} \right] \quad 1 + \frac{2}{|y-2|}$$

$$\downarrow \quad x^2 - 2(y+1)^2 = 20$$

$$x = -4, \dots, 4$$

$$y = -6, \dots, 2$$

$$x^2 - 2y^2 - 4y - 8 = -4, \dots, 4$$

Teorema

Sia $F_0(x, y)$ la parte omogenea di grado massimo di F . Se F_0 ha almeno due fattori irr. distinti su $\mathbb{Q}$, allora la strategia funziona

$$x^2 y^2 - y^2 + xy - x^3 + 1 = 0$$

$$(x^2 - 1)y^2 + xy - x^3 + 1 = 0$$

$$\Delta = x^2 + 4(x^2 - 1)(x^3 - 1) = 4x^5 - 4x^3 - 3x^2 + 4$$

$$y_{\pm} = \frac{-x \pm \sqrt{\Delta}}{2(x^2 - 1)}$$

$$\text{Per } x \leq -2 \\ \Delta < 0$$

$$\begin{aligned} \sqrt{\Delta} &= \sqrt{4x^5 - 4x^3 - 3x^2 + 4} = 2x^{5/2} \sqrt{1 - \frac{1}{x^2} - \frac{3}{4x^3} + \frac{1}{x^5}} \\ &= 2x^{5/2} \left(1 - \frac{1}{2x^2} - \frac{3}{8x^3} - \frac{1}{2x^5} + O\left(\frac{1}{x^4}\right) \right) \end{aligned}$$

$$\sqrt{1+x} \underset{x \rightarrow 0}{=} 1 + \frac{x}{2} + O(x^2)$$

$$\frac{1}{x^2 - 1} = \frac{1}{x^2} \frac{1}{1 - 1/x^2} = \frac{1}{x^2} \left(1 + \frac{1}{x^2} + O\left(\frac{1}{x^4}\right) \right) = \frac{1}{x^2} + \frac{1}{x^4} + O\left(\frac{1}{x^6}\right)$$

$$y_{\pm} = \frac{1}{2} \left(\frac{1}{x^2} + \frac{1}{x^4} + O\left(\frac{1}{x^6}\right) \right) \left(2x^{5/2} - x - x^{1/2} - \frac{3}{4}x^{-1/2} + O(x^{-3/2}) \right)$$

$$y_{+} = x^{1/2} \left(-\frac{1}{2x} \right) + \frac{1}{2} x^{-3/2} - \frac{3}{8} x^{-5/2} - \frac{1}{2x^3} + O(x^{-7/2})$$

$$y_{-} = -x^{1/2} \left(-\frac{1}{2x} \right) - \frac{1}{2} x^{-3/2} + \frac{3}{8} x^{-5/2} - \frac{1}{2x^3} + O(x^{-7/2})$$

$$y_{\pm} + \frac{1}{2x} = \pm \sqrt{x} \left(1 + \frac{1}{2} \frac{1}{x^2} - \frac{3}{8} \frac{1}{x^3} + O\left(x^{-7/2}\right) \right)$$

$$(2xy + 1)^2 = 4x^3 \left(1 + \frac{1}{x^2} - \frac{3}{4} \frac{1}{x^3} + O(x^{-7/2}) \right)$$

$$\underbrace{(2ny+1)^2 - 4n^3 - 4n + 3 = 0}_{4n^2y^2 + 4ny + 1 - 4n^3 - 4n + 3} \quad (n^{-1/2})$$

$$4n^2y^2 + 4ny + 1 - 4n^3 - 4n + 3$$

$$\boxed{y^2 - n}$$

$$(y^2 - n)(n^2 - 1) = n^2y^2 - 3n^3 - y^2 + n =$$

$$= n - ny - 1$$

$$|y^2 - n| = \left| \frac{ny - n + 1}{n^2 - 1} \right|$$

Our assumption
 $n \geq 2$

$$|y_+| \leq |y_-|$$

$$|y^2 - n| \leq \left| \frac{ny_-}{n^2 - 1} \right| + \frac{1}{n+1}$$

$$\Delta \leq 4n^5$$

$$|y_{\pm}| \leq \frac{2n^{5/2} + n}{2(n^2 - 1)}$$

$$|y^2 - n| \leq \frac{2n^{7/2} + n^2}{2(n^2 - 1)^2} + \frac{1}{n+1}$$

$\bar{e}$ decreases

im $\bar{e} < 2$

$$\leq \frac{26}{9} \frac{1}{n^{4-2}} \frac{(2n^{7/2} + n^2)}{n^2} + \frac{1}{n+1} = \frac{8}{9} \frac{3n^{7/2}}{n^4} + \frac{1}{n+1}$$

$$= \frac{8}{3} \frac{1}{5n} + \frac{1}{n+1} \quad \text{im } \bar{e} < 3$$

$$x = -1, 0, 1 \quad |y^2 - x| = 0, 1$$

$$x = -1, 0, 1, \quad x = y^2, \quad x = y^2 + 1, \quad x = y^2 - 1$$

$$x = -1 \quad -y + 1 + 1 = 0 \quad y = 2 \quad (-1, 2)$$

$$x = 0 \quad -y^2 + 1 = 0 \quad (0, \pm 1)$$

$$x = 1 \quad y - 1 + 1 = 0 \quad (1, 0)$$

$$x = y^2 \quad \cancel{y^3 - y^2 + 1 = 0} \quad \times$$

$$x = y^2 \pm 1 \quad \mp [(y^2 \pm 1)^2 - 1] + (y^2 \pm 1)y - y^2 \mp 1 + 1 = 0$$

$$y^4 + 2y^2 - y^3 - y + y^2 = 0$$

$$\mp (y^4 \pm 2y^2) + y^3 \pm y - y^2 + 1 \mp 1 = 0$$

$$-y^4 + 2y^2 + y^3 + y - y^2 = 0 \quad y = 0 \quad \times$$

$$y^4 - 2y^2 + y^3 - y - y^2 + 2 = 0$$

$$-y^3 + 2y + y^2 + 1 - y \quad \times$$

$$y^4 + y^3 - 3y^2 - y + 2 = 0$$

$$16 + 8 - 12 - 2 + 2 \quad \times$$

$$16 - 8 - 12 + 2 + 2 \quad \checkmark$$

$$y = 1 \quad 0$$

$$y = -1 \quad 0$$

$$y = -2 \quad \}$$

$$x = \boxed{3, -2}$$

$$x^4 - y^4 = x^2 y + 2xy + x - y - 3$$

$$x^3 y^3 = x^4 + y^4 + 32$$

$$y^2 = \frac{x^4 + 2x^3 - x^2 + 3x - 1}{(x^2 + ax + b)^2}$$

$$(x^2 + ax + b)^2$$

$$x^4 + 2ax^3 + (a^2 + 2b)x^2 + 2abx + b^2$$

$$\begin{array}{c} || \\ 2 \\ a=1 \end{array}$$

$$\begin{array}{c} || \\ -1 \\ b=-1 \end{array}$$

$$\begin{array}{c} || \\ -2x \\ +1 \end{array}$$

$$y^2 = (x^2 + x - 1)^2 + 5x - 2$$

$$(y + x^2 + x - 1)(y - x^2 - x + 1) = 5x - 2$$

$$y \sim x^2$$



$$1 < \frac{|5x-2|}{|x^2+x-1|}$$



$$y^n = \frac{ax^d + \dots}{\dots}$$

Se

$d > 1$ un divisore comune di n e d

a è una potenza di esima

allora la strategia funziona

$$y^3 = x^6 + \dots$$

Il teorema di Minkowski

$\mathbb{R}^n$

Un RETICOLO in $\mathbb{R}^n$ è un insieme

$$\Lambda = \left\{ \sum_{i=1}^n c_i v_i \mid c_i \in \mathbb{Z} \right\}$$

$v_i \in \mathbb{R}^n$

indipendenti

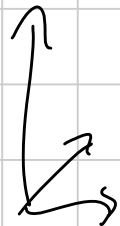
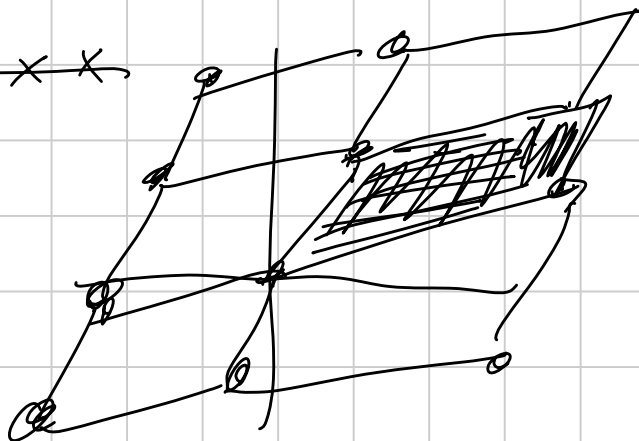
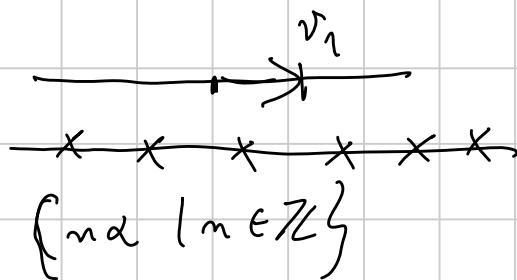


non esiste un'equazione

$$a_1 x_1 + \dots + a_n x_n = 0$$

con $a_i \in \mathbb{R}$ risolta da tutti

i $v_1, \dots, v_n$



Il ~~dominio~~ dominio fondamentale
di un reticolo Λ generato da
 $v_1, \dots, v_n$

$$P = \left\{ \sum_{i=1}^n c_i v_i \mid 0 \leq c_i < 1 \right\}$$

- Se $v, w \in P$ allora $v - w \notin \Lambda$
- $\forall v \in \mathbb{R}^n \exists w \in \Lambda \mid v - w \in \Lambda$

$$v = \sum a_i v_i \quad 0 \leq a_i, b_i < 1$$

$$w = \sum b_i v_i$$

$$v - w = \sum (a_i - b_i) v_i$$

$\downarrow$
 $1 - 1 = 0 < 1$

$$v = \sum a_i v_i \quad b_i = \lfloor a_i \rfloor$$

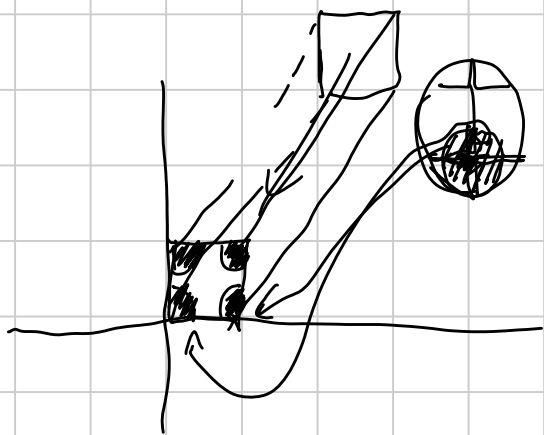
$$w = \sum b_i v_i \in \Lambda$$

$$v - w = \sum (a_i - b_i) v_i \in P$$

$\uparrow$
 $0 \leq < 1$

$$p: \mathbb{R}^n \rightarrow P$$

$v \mapsto w$ l'unico elemento di P
tale che $v - w \in \Lambda$



p preserva localmente
i volumi

$$\text{Se } U \subseteq \mathbb{R}^n$$

$$\text{tale che } \forall v, w \in U$$

$$v - w \notin \Lambda$$

allora $p(U)$ e U hanno lo stesso
volume

$$U = \bigcup_{v \in \Lambda} U \cap (P + v)$$

$$\text{se } \exists w_1, w_2 \in U$$

$$\text{tale che } p(w_1) = p(w_2)$$

$$w_1 - v_1 = p(w_1) = p(w_2) = w_2 - v_2 \in \Lambda$$

$$w_1 - w_2 = v_1 - v_2 \in \Lambda$$

Teorema

Sia

$K \subseteq \mathbb{R}^n$ di volume $\text{vol}(K)$

- connesso
- simmetrico-centro

$\Lambda \subseteq \mathbb{R}^n$ reticolo di volume $\text{Vol}(\Lambda)$

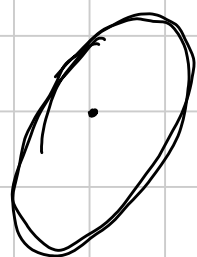
$$\bullet \forall x, y \in K \quad x \in \{ \lambda x + (1-\lambda)y \mid \lambda \in [0,1] \}$$

$$\forall x \in K \quad -x \in K$$

1°
K

$$\text{Se } \text{Vol}(K) > 2^n \cdot \text{vol}(\Lambda)$$

allora K contiene un elemento di Λ
 $\neq 0$



K

$P($

$$K' = \frac{1}{2} K = \left\{ \frac{x}{2} \mid x \in K \right\}$$

$$\text{vol}(K') = \frac{\text{vol} K}{2^n}$$

$P(K')$

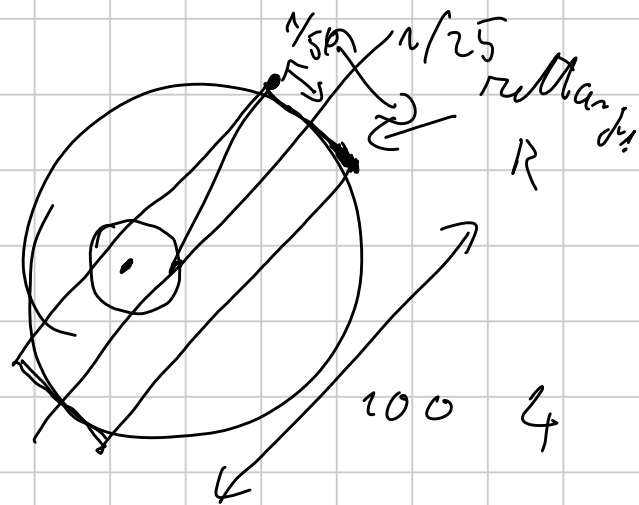
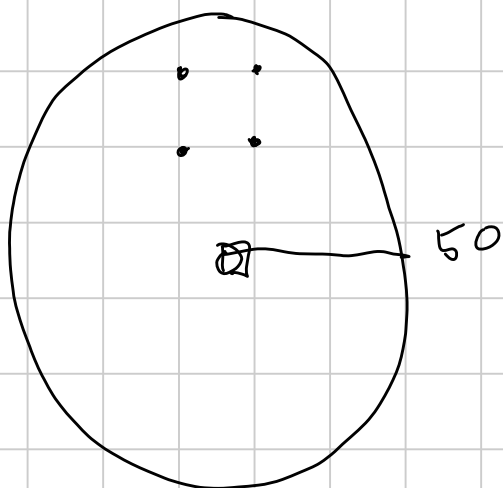
$$\text{vol}(K') > \text{vol}(\Lambda)$$

$$\text{vol}(P(K')) \leq \text{vol}(\Lambda)$$

$$\Rightarrow \exists x, y \in K' \text{ tali che } x - y \in \Lambda$$

$$2x \quad 2y \in K \quad -2y \in K \quad \frac{2x - 2y}{2} \in K$$

Λ



~~50~~
se

$$\exists v \in \mathbb{Z}^2 \cap R$$

$$|v| > 50 \quad \text{allora } |v| \geq \sqrt{50^2 + 1}$$

il rettangolo descritto rettangolo di lato $\sqrt{50^2 + 1}$

Sia p primo dispari $p \equiv 1 \pmod{4}$

$$\left(\frac{-1}{p}\right) = 1 \iff \exists n \in \mathbb{Z} \mid n^2 \equiv -1 \pmod{p}$$

$$t \text{ generatore} \quad t^{\frac{p-1}{4}} = n \quad n^2 = t^{\frac{p-1}{2}} \equiv -1$$

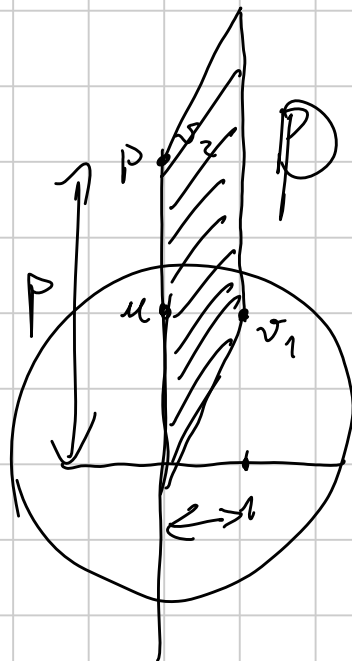
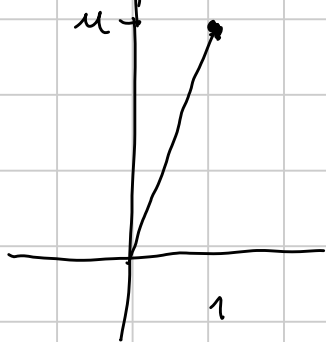
Sia $u \in \mathbb{Z}$ tale che $u^2 \equiv -1 \pmod{p}$

$$\Lambda = \left\{ (a, b) \in \mathbb{Z}^2 \mid b \equiv ua \pmod{p} \right\}$$

è un reticolo

$$\begin{pmatrix} 1, u \\ 0, p \end{pmatrix} \in \Lambda$$

$$P \mid (a, b) \in \Lambda = \text{or } (a, b) = a(1, u) + \frac{b-ua}{P}(0, P)$$



$$\text{vol}(\Lambda) = P$$

mostro

$$K = \text{cerchio di raggio } r \quad \text{vol } K = \pi r^2$$

$$\text{Se } \underbrace{\pi r^2}_{\text{vol}(K)} \geq 4P = 2^2 \cdot \text{vol}(\Lambda)$$

$$\text{allora } \exists v \in \Lambda \quad v \neq 0 \quad |v| \leq r$$

$$r = 2\sqrt{\frac{P}{\pi}} \quad \exists (a, b) \in \mathbb{Z}^2$$

$$\text{tali che } a \equiv ub \quad \sqrt{a^2 + b^2} \leq \frac{2}{\sqrt{\pi}} \sqrt{P}$$

$$(a, b) \neq (0, 0)$$

$$a^2 + b^2 \leq \frac{4}{\pi} P < 2P$$

$$0 < a^2 + b^2 \equiv u^2 b^2 + b^2 \equiv -b^2 + b^2 \equiv 0 \pmod{P}$$

$$\Rightarrow a^2 + b^2 = P$$

$$(x^2 + y^2)(a^2 + b^2) = (ax - by)^2 + (xb + ya)^2$$

||

$$|| \quad |n+iy|^2 \cdot |a+ib|^2 = |(n+iy)(a+ib)|^2 = |an-by+i(nb+ay)|^2$$

$n \in \mathbb{N}_{>0}$ è somma di due quadrati
 se e solo se, nella fattorizzazione di n ,
 tutti i primi $\equiv 3 \pmod{4}$ compaiono con
 esponente pari