

A2 - BASIC

Note Title

03/09/2026

Notazione $\sum_{i=1}^n a_i$

• Somma ciclica $\sum_{cyc} f(a, b, c)$

• " Simmetrica $\sum_{sym} f(a, b, c)$

$$\sum_{cyc}^1 a^2 b = a^2 b + b^2 c + c^2 a$$

$\overset{1}{a^2 b \cdot c}$

$$\sum_{cyc} abc = 3abc$$

$$\sum_{sym} abc = 6abc$$

$$\sum_{sym}^1 a^2 b = a^2 b + a^2 c + b^2 a + b^2 c + c^2 a + c^2 b$$

$$\rightarrow \square \geq 0 \quad \text{es: } a^2 + b^2 \geq 2ab \Leftrightarrow (a-b)^2 \geq 0$$

$\Rightarrow a=b$

cas: di uguaglianza
da verificare

$$a^2 + b^2 \geq 2ab$$

TdA 2024: $\underbrace{(2+3a)}_{x+y} \underbrace{(1+2b)}_{x=y} \underbrace{(3a+4b)}_{x=y} = \underline{ab \cdot 96} \quad a, b > 0$

$$(2+3a) \geq \underbrace{(2\sqrt{6a})}$$

$2=3a \quad 1=2b \quad 3a=4b$

Cauchy-Schwarz

$$a_1, a_2, \dots, a_n \in \mathbb{R} \Rightarrow$$
$$b_1, b_2, \dots, b_n \in \mathbb{R}$$

$$\boxed{\left(\sum a_i^2\right) \left(\sum b_i^2\right)} \geq \boxed{\left(\sum a_i b_i\right)^2}$$

$$eS \quad n=2$$

$$a_1, a_2$$

$$b_1, b_2$$

$$\Rightarrow (a_1^2 + b_1^2)(a_2^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$$

$$\frac{a_i}{b_i} = \text{cost}$$

• Nesbitt $\left\| \sum_{cyc} \frac{a}{b+c} \right\| \geq \frac{3}{2} \quad a, b, c > 0$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$\left(\sum x_i^2 \right) \geq \frac{\left(\sum x_i y_i \right)^2}{\sum y_i^2}$$

• 1° tentativo

$$x_1 = \sqrt{\frac{a}{b+c}}$$

$$x_2 = \sqrt{\frac{b}{a+c}}$$

$$x_3 = \sqrt{\frac{c}{a+b}}$$

$$y_1 = \sqrt{(b+c) \cdot a}$$

$$\frac{(a+b+c)^2}{(b+c) + (a+c) + (a+b)}$$

$$\geq \frac{(a+b+c)^2}{2(ab+bc+ac)} \geq \frac{3}{2}$$

$$\boxed{a^2 + b^2 + c^2 \geq ab + bc + ac}$$

$$a^2 + b^2 \geq 2ab$$

$$b^2 + c^2 \geq 2bc$$

$$a^2 + c^2 \geq 2ac$$

→ per le somme cicliche

Scrivere al contrario

$$a_i = \frac{x_i}{\sqrt{y_i}} \quad \text{e} \quad b_i = \sqrt{y_i}$$

$$\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i} \rightarrow \text{disu di Titu}$$

$$x_1 = a \quad y_1 = a(b+c)$$

$\Rightarrow$ Medie

$$a_i > 0$$

$$\frac{\sum a_i}{n} \geq \sqrt[n]{\prod a_i}$$

$$n \Rightarrow \infty \quad n \Rightarrow n-1$$

$$\frac{a+b}{2} \geq \sqrt[3]{ab}$$

es: Trovare il max di $x^5 y^2 z$ con $x+y+z=1$

$$\frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{y}{2}, \frac{y}{2}, z$$

$$5x + 2y + z$$

$$\frac{x+y+z}{8} \geq \sqrt[8]{\frac{x^5 y^2 z}{5^5 \cdot 2^2}}$$

$$\bullet \quad a_i > 0 \quad p_i > 0 \quad \sum p_i = 5$$

$$\Rightarrow \frac{1}{5} \sum a_i p_i \geq \sqrt[5]{\prod a_i^{p_i}} \rightarrow \text{AM-GM pesate}$$

• Media p -esima

$$M_p := \sqrt[p]{\frac{\sum a_i^p}{n}} =$$

$$= \sqrt[p]{\frac{a_1^p + a_2^p + \dots + a_n^p}{n}}$$

$$p=1 \quad \text{AM}$$

$$p=2 \quad \text{QM}$$

$$p=-1 \quad \text{HM}$$

$$HM = \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$$

$$p \rightarrow 0 \quad GM$$

$$\text{Se } p \geq q \Rightarrow M_p \geq M_q$$

Se tutti uguali

↳ caso di uguaglianza

$$p \rightarrow \pm \infty$$

$$\max\{a_i\} M_p \geq \min\{a_i\}$$

→ problema: Taiwan TST-2014

$$AOPS \quad \downarrow \quad a, b, c > 0 \text{ dim}$$

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} + 8\sqrt[3]{abc} \leq 3(a+b+c)$$

$$M_p \quad p=3 \quad p=0 \quad p=-3 \quad p=\frac{1}{3} \quad p=-\frac{1}{3}$$

$$M_{\frac{1}{3}} = \left(\frac{x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} + \dots + x_n^{\frac{1}{3}}}{n} \right)^3$$

$$\frac{a^3+b^3+c^3}{3} = x_1$$

$$abc = x_2$$

$$abc = x_3$$

$$\vdots$$

$$abc = x_9$$

$$\left(M_{\frac{1}{3}} \left(\frac{a^3+b^3+c^3}{3}, \underbrace{abc, abc, \dots, abc}_{8 \text{ volte}} \right) \right)^{\frac{1}{3}} =$$

$$= M_{\frac{1}{3}} \leq M_1 = \frac{a^3+b^3+c^3}{3} + 8abc$$

$$\left(M_{\frac{1}{3}} \right)^{\frac{1}{3}} \leq \left(\frac{\frac{a^3+b^3+c^3}{3} + 8abc}{9} \right)^{\frac{1}{3}} \leq 3(a+b+c)$$

$$3(a^3+b^3+c^3+24abc) \leq 3(a+b+c)^3$$

$$\Leftrightarrow a^3+b^3+c^3+24abc \leq (a+b+c)^3$$

$$\Leftrightarrow a^3 + b^3 + c^3 + 2abc \leq a^3 + b^3 + c^3 + 6abc + 3 \sum_{\text{sym}} a^2b$$

$$\Leftrightarrow 6abc \leq \sum_{\text{sym}} a^2b \quad \frac{\sum_{\text{sym}} a^2b}{6} \geq \sqrt[6]{a^6 b^6 c^6}$$

Bunching / Muirhead

$$a^2 + b^2 \geq 2ab \quad \sum_{\text{sym}} a^2 b^0 > \sum_{\text{sym}} a^1 b^1$$

$$a_i > 0 \quad x_i, y_i \in \mathbb{R}$$

termini di
grado uguale

Se vale che $\begin{cases} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \\ \sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i \quad \forall k \in \{1, \dots, n\} \end{cases}$

↳ "concentrazione"

$$\Rightarrow \sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \dots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \dots a_n^{y_n}$$

$$(2, 0, 0) > (1, 1, 0) \quad \text{ordine conto obv } \triangle$$

! Bunching solo x somme

simmetriche, non cicliche

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a \quad \text{no Bunching}$$

Riarrangiamento

$$\{1, 2, 3, 4\} \quad \{3, 7, 9, 10\}$$

a_i b_i quando max e min
della somma dei prodotti di coppie

• Siamo $a_1 \leq a_2 \leq \dots \leq a_n$ $b_1 \leq b_2 \leq \dots \leq b_n$

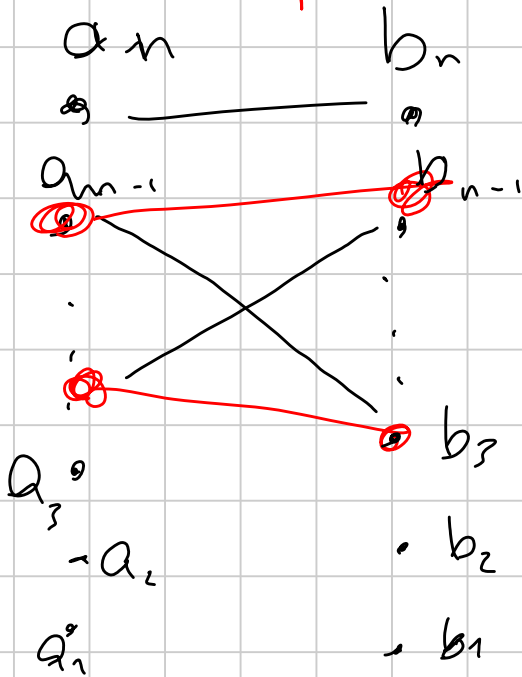
$$\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

$\sum a_i b_{\sigma(i)}$ è massima se $\sigma(i) = i$
è minima se $\sigma(i) = n+1-i$

$$n=2$$

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1$$

$$(a_1 - a_2)(b_1 - b_2) \geq 0$$



a_i lo trasforma in $a_i + t$

b_i " $b_i + s$

$$\sum (a_i + t)(b_{\sigma(i)} + s) = \left[\sum a_i b_{\sigma(i)} \right] + s \sum a_i + t \sum b_{\sigma(i)} + st \cdot n$$

Omogenea

$$\text{def: } f(\lambda a, \lambda b, \lambda c) = f(a, b, c)$$

f omogenea se $\forall \lambda \neq 0$

$$f = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \quad \checkmark \quad f = a+b+c \quad \times$$

$$f(a, b, c) = \frac{a}{b} \quad \checkmark \quad f(x, y) = \frac{xy}{x^2+y^2} \quad \checkmark$$

$$\text{es: Se } abc=1 \quad a^2+b^2+c^2 \stackrel{?}{\geq} a+b+c$$

$$a, b, c > 0$$

$$a^2+b^2+c^2 \stackrel{?}{\geq} \sqrt[3]{abc} (a+b+c)$$

$$a^2+b^2+c^2 \stackrel{?}{\geq} a^{\frac{2}{3}} \cdot b^{\frac{1}{3}} \cdot c^{\frac{1}{3}} + a^{\frac{1}{3}} b^{\frac{2}{3}} c^{\frac{1}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} c^{\frac{2}{3}}$$

$(2, 0, 0) \quad (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$

$$(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$$

$$\sum a_i^2 = 1 \quad \sum b_i^2 = 1$$

$$(\sum \overbrace{(a_i b_i)})^2 \leq 1$$

$$a_i b_i \leq \frac{a_i^2 + b_i^2}{2}$$

$$\sum a_i b_i \leq \frac{\sum a_i^2}{2} + \frac{\sum b_i^2}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$a+b+c=1 \quad a+2b+4c=1 \quad a^2+b^3+c^4=1$$

$\times$

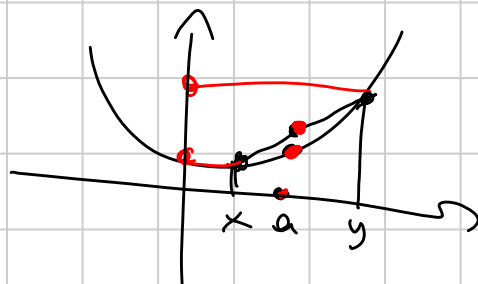
def: funzione convessa $f: \mathbb{R} \rightarrow \mathbb{R}$

$$\forall x, y \quad t \in [0, 1] \quad f(\underline{tx + (1-t)y}) \leq tf(x) + (1-t)f(y)$$

$$tx + (1-t)y \quad t \in [0, 1]$$

$$f(a)$$

$$f'' \geq 0$$

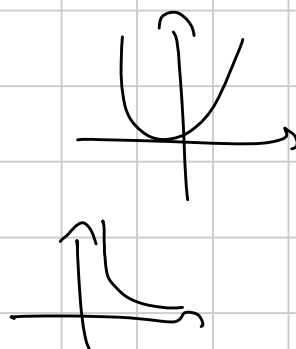


funzioni continue basta $t = \frac{1}{2}$

es: x^2 è convessa? sì

$$\left(\frac{x}{2} + \frac{y}{2}\right)^2 \leq \frac{x^2 + y^2}{2}$$

$\frac{1}{x}$ è convessa



$\sqrt{x}$ è concava

$-f$ è convessa



$\Downarrow$
f concave

log è concava

Jensen

$f: (a, b) \rightarrow \mathbb{R}$ è convessa

$$x_1, \dots, x_n \in (a, b) \Rightarrow f\left(\sum t_i x_i\right) \leq \sum t_i f(x_i)$$

$$t_i \in [0, 1] \quad \sum t_i = 1 \quad t_i = \frac{1}{n} \quad \boxed{f\left(\frac{\sum x_i}{n}\right) \leq \frac{\sum f(x_i)}{n}}$$

es: $f(x) = \frac{1}{x} \quad \left(\frac{1}{n} \cdot \sum x_i\right)^{-1} \leq \left(\sum \frac{1}{x_i}\right) \cdot \frac{1}{n}$

$\frac{1}{x_i} = y_i \quad \frac{n}{\sum \frac{1}{y_i}} \leq \frac{\sum y_i}{n} \quad AM \geq HM$

Jensen con logaritmo

x ottenere AM-GM

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2} \quad \sum_{cyc} f(a)$$

vorriamo ricondurci a funzione in 1 variabile, e' enough $\Rightarrow$ uncelo

$$a+b+c=1 \quad \sum_{cyc} \frac{a}{1-a} = \sum_{cyc} \left(\frac{1}{1-a} - 1 \right) \stackrel{?}{\geq} \frac{3}{2}$$

$$\Rightarrow \sum_{cyc} \frac{1}{1-a} \geq \frac{3}{2} \quad a, b, c \quad \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$$

$$f(x) = \frac{1}{1-x} \quad \left(\frac{1}{1-\frac{a+b+c}{3}} \right) \leq \frac{f(a)+f(b)+f(c)}{3}$$

$$= \left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \right) \cdot \frac{1}{3}$$

$$\Rightarrow \frac{3}{2} \leq \left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \right) \cdot \frac{1}{3} \Rightarrow \text{No! it}$$

$$\boxed{a_{n+2} \geq a_n + 3} \Rightarrow a_{2023} \geq 3034$$

$$a_{n+2} = \sqrt{\left(\sum_{i=1}^n x_i + (x_{n+1} + x_{n+2}) \right) \left(\sum_{i=1}^n \frac{1}{x_i} + \frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right)} \stackrel{C-S}{\geq}$$

$$\geq \sqrt{\left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n \frac{1}{x_i} \right)} + \sqrt{(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right)} =$$

$$\sqrt{\sum_{i=1}^n x_i} = a_1 \quad a_2 = \sqrt{x_{n+1} + x_{n+2}}$$

$$b_1 = \sqrt{\sum_{i=1}^n \frac{1}{x_i}} \quad b_2 = \sqrt{\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}}}$$

$$\left[\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} \geq \left(\sum_{i=1}^n a_i b_i \right)^2 \right]$$

$$= a_n + \sqrt{(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right)} \geq 2$$

Lemma 1 $\left(\sqrt{(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right)} \geq 2 \right)$

$$\Leftrightarrow (x_{n+1} + x_{n+2}) \geq 2 \sqrt{x_{n+1} x_{n+2}}$$

$$a^2 + b^2 \geq 2ab$$

$$a = \sqrt{x_{n+1}} \quad b = \sqrt{x_{n+2}}$$

$$(a, b, c) = (a, c, b) \quad k_1 = \frac{a}{a+c} + \frac{c}{b+c} + \frac{b}{a+b} \leq k_2$$

$$\left(\frac{a}{a+c} + \frac{c}{b+c} + \frac{b}{a+b} \right) + \left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \right) = 3$$

$$k_1 + k_2 = 3$$

$$E_2$$

$$k_2 = 3 - k_1$$

$$E_1 + E_2 = 3$$

$$E_1 = 3 - E_2$$

$$k_1 = 1$$

$$\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i}$$

$$\sum_{i=1}^n \frac{a_i}{a_i + b_i} \geq \frac{\sum a_i^2}{\sum a_i^2 + \sum a_i b_i} \geq \frac{(\sum a_i)^2}{\sum a_i^2 + \sum a_i b_i} = \frac{\sum a_i^2 + 2 \sum_{i < j} a_i a_j}{\sum a_i^2 + \sum_{i < j} a_i b_i} > 1$$

Costruzione $(1, \varepsilon, \varepsilon^2) = (a, \overset{c}{b}, b)$

$$\frac{1}{1+\varepsilon^2} + \frac{\varepsilon^2}{\varepsilon+\varepsilon^2} + \frac{\varepsilon}{1+\varepsilon} = \frac{1}{1+\varepsilon^2} + \frac{2\varepsilon}{1+\varepsilon} \quad \varepsilon \rightarrow 0$$