

A3-BASIC

Note Title

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$$\rightarrow 1 + x + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n x^i = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

$|x| < 1$

$$\rightarrow 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$1 + 4 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1 + 8 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Succession:

$$\begin{cases} a_{n+1} = f(n, a_0, \dots, a_n) \\ a_0, a_1, \dots, a_n \end{cases} \quad f(n) = n^2 \cdot 2^n$$

$$\begin{aligned} a_{n+1} &= a_n + n^2 \\ a_{n+1} &= a_n^2 + a_n^3 + 3 \end{aligned}$$

es: $a_{n+1} = 3a_n + 2$

$$1, 3, 9, 27, \dots \rightarrow a_n = 3^n$$

$$1, 5, 17$$

$$a_n = c$$

$$b_n = a_n + c \rightarrow a_n = b_n - c \quad a_{n+1} = b_{n+1} - c$$

$$\begin{cases} b_{n+1} - c = 3b_n - 3c + 2 \\ b_0 = a_0 + c = 2 \end{cases} \quad (c = +1)$$

$$b_0 = a_0 + c = 2$$

$$b_n = 2 \cdot 3^n \rightarrow a_n = 2 \cdot 3^n - 1$$

$$a_{n+1} = \sum \alpha_i a_i \quad \text{con } \alpha_i \text{ constanti}$$

$$a_{n+1} = 3a_n^2$$

$$a_{n+1} = 3a_n \cdot a_{n-1}$$

$$a_{n+1} = n \cdot a_n$$

no lineari

$$\begin{cases} a_{n+2} = a_n + a_{n+1} \\ a_0, a_1 = 1 \end{cases}$$

$$a_n = \lambda^n$$

$$\lambda^{n+2} = \lambda^n + \lambda^{n+1}$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$S_1(n+2) = S_1(n) + S_1(n+1)$$

→ Teorema

$$a_n = S_1(n) \quad a_n = S_2(n)$$

$$a_n = A \cdot S_1(n) + B S_2(n) \quad \text{è soluzione}$$

$$(\cancel{A \cdot S_1(n+2)}) + B \cancel{S_2(n+2)} = \cancel{A S_1(n)} + \cancel{B S_2(n)} \quad \cancel{A S_1(n+1)} + \cancel{B S_2(n+1)}$$

→ Teorema (misturboso)

Le soluz. lin. indipendenti sono k
dove k è da quanti termini prima la
ricorrenza dipende

$$\left(\text{es. } a_{n+2} = a_{n+1} + a_n, \quad \lambda^{n+2} = \lambda^{n+1} + \lambda^n \right)$$

$$\lambda^3 = \lambda^2 + 1$$

polinomio associato

$$a_n = A \left(\frac{1+\sqrt{5}}{2} \right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\lambda = 1$$

$$a_n = 1^n \cdot A + 1^n \cdot n \cdot B$$

$$\text{es: } \begin{cases} a_{n+2} = a_{n+1} + a_n + 2 \\ a_0, a_1 = 1 \end{cases}$$

$$\lambda^{n+2} = \lambda^{n+1} + \lambda^n + 2$$

es: $\begin{cases} a_{n+2} = 2a_{n+1} + a_n + 2n \\ a_0, a_1, \text{ dati} \end{cases} \quad a_n = -n$

$-n-2 = -2n = 2-n+2n$

$A\lambda^{n+2} + B\lambda^{n+2} - n-2 = 2A\lambda^{n+1} + 2B\lambda^{n+1} - 2n + A\lambda^n + B\lambda^n - n + 2n$

$(a_n = A\lambda^n + B\lambda^n - n)$

Funktionen

$f(x) = c \cdot x$

$f: \mathbb{R} \rightarrow \mathbb{R} \quad f: \mathbb{Q} \rightarrow \mathbb{Q}$

$(f(x) + f(y) = f(x+y))$

$f(x^2) + f(yf(x)) = f(y) + x$

Sketch

Le uniche soluzioni sono $f(x) = \dots$, in fatti:

$f(x) + f(y) = x + y = f(x+y)$

Dimostrate che sono le uniche

→ valori particolari $x=0$

$x=y$

→ guardare iniettività / suriettività

NOTA

$f(x f(x) + f(y)) = f(x)^2 + y \quad f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = x$
 $f(x) = -x$

$x=0$

$f(f(y)) = f(0)^2 + y$

Se $f(n) = y + k$

⇒ pensare alle suriettività

⇒ f suriettiva

(Achtung! $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$)

↙ $\exists t \mid f(t) = 0 \leadsto$ potentissimo

$$x=t \quad f(\cancel{t} f(t) + f(y)) = 0^2 + y \leadsto f(f(y)) = y$$

$$\Rightarrow f(0) = 0$$

Standard è dire prendo $a, b \mid f(a) = f(b)$

$$\underbrace{f(x f(x) + f(y)) = f(x)^2 + y}$$

fissare x

e mette $a=y$
 e poi $b=y$

$$f(x)^2 + a = f(x)^2 + b \Rightarrow a = b$$

$$f(\text{qualcosa } 1) = f(\text{qualcosa } 2)$$

$$\Rightarrow \text{qualcosa } 1 = \text{qualcosa } 2$$

iniettività "semplificare f "

$f(g(x))$ è iniettiva $\Rightarrow g(x)$ è iniettiva

$$f(g(a)) = f(g(b)) \quad g(a) = g(b)$$

$f(g(x))$ è suriettiva $\Rightarrow f$ è suriettiva

$$f(x f(x) + f(y)) = f(x)^2 + y$$

$$f(f(x)) = x$$

f f trick

$$P(f(x), y)$$

$$f(f(x) \cdot f(f(x)) + f(y)) = f(f(f(x)))^2 + y$$

$$f(f(x) \cdot x + f(y)) = x^2 + y$$

$$\Rightarrow f(x)^2 + y = x^2 + y$$

$$f(x) = \pm x$$

$$x \Rightarrow f(x)$$

$$f(x) = \sim$$

ATTENZIONE!

$$y = f(x)$$

$$f(y) = \sim$$

$$f(x) = \sim$$

$$f(a) = a \quad f(b) = -b \quad a, b \neq 0$$

$$x=a \quad y=b \quad f(a^2-b) = a^2+b$$

$$\text{es: } f(x^2+y) = f(x^3+2y) + f(x^4) \quad f: \mathbb{R} \rightarrow \mathbb{R}$$

$$x=0 \quad f(y) = f(2y) + f(0) \quad f(x) = 0$$

$$\text{poniamo } x^2+y = x^3+2y \Rightarrow y = x^2 - x^3$$

$$f(x^4) = 0 \quad f(x) = 0 \quad \text{se } x \geq 0$$

$$y=0 \quad \cancel{f(x^2)} = f(x^3) + f(x^4) \Rightarrow f(x^3) = 0 \quad f(x) = 0 \quad \forall x$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{es: } f(x(x+f(y))) = (x+y)f(x) \quad f(x) = 0$$

$$f(x) = x$$

$$P(0, y) \quad f(0) = yf(0) \Rightarrow f(0) = 0$$

$$P(x, 0) \quad f(x^2) = xf(x)$$

$$P(x, -x) \quad \boxed{f(x(x+f(-x))) = 0} \quad (1)$$

cerchiamo di ottenere l'identità

$$a, b \quad f(a) = f(b)$$

$$a=y, \quad b=y$$

$$f(x(x+f(a))) = (x+a)f(x) \quad (x+a)f(x) = (x+b)f(x)$$

$$f(x(x+f(b))) = (x+b)f(x)$$

$$\text{Se } f(x) \neq 0 \Rightarrow a=b \quad \text{Sendo } f(x) = x \quad \forall x$$

$\Rightarrow$ nella (1) posso "semplificare" f

$$x(x+f(-x)) = 0 \Rightarrow f(-x) = -x \quad f(x) = x$$

Cauchy $f(x) = cx$

$$f(x) + f(y) = f(x+y)$$

$$f: \mathbb{Z} \rightarrow \mathbb{Z} \quad f(1)$$

$$2f(x) = f(2x)$$

$$x=y \Rightarrow$$

$$2f(6) = f(6)$$

$$\Rightarrow f(6) = 0$$

$$y=1 \quad f(x+1) = f(x) + 1 \Rightarrow f(n) = n \cdot f(1)$$

$$f(x+1) - f(x) = \text{constante}$$

$$f: \mathbb{Q} \rightarrow \mathbb{Q}$$

$\Downarrow$
f lineare

$$f\left(x + \frac{1}{n}\right) = f(x) + f\left(\frac{1}{n}\right) \quad f\left(\frac{m}{n}\right) = m f\left(\frac{1}{n}\right)$$

$$f(1) = n \cdot f\left(\frac{1}{n}\right) \Rightarrow f\left(\frac{m}{n}\right) = \frac{m}{n} f(1)$$

$$\begin{cases} f(x)f(y) = f(xy) \\ f(x)f(y) = f(x+y) \\ f(x) + f(y) = f(xy) \end{cases}$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

Continuität, monotonie, ...

1) $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(f(x)^2 + f(y)) = x f(x) + y$$

2) $f: \mathbb{Z} \rightarrow \mathbb{Z}$

$$f(2a) + 2f(b) = f(f(a+b))$$

3) $X_{n+1} = 6X_n + 2 \sum_{i=0}^n X_i \quad X_0 = 1$

4) $f(x + f(y)) = f(x) + y \quad f: \mathbb{Q} \rightarrow \mathbb{Q}$

$$1) f(f(x)^2 + f(y)) = \underline{f(x) \cdot x + y} \quad f(x) = x \quad f(x) = -x$$

fixare x e far variare $y \Rightarrow f$ surjective

$$a=y \quad b=y \quad f(a)=f(b) \quad \underline{f(x) \cdot x + a = f(x) \cdot x + b}$$

$\Rightarrow$ injective

$$\forall t \quad f(t) = 0$$

$$P(t, y) \quad f(f(y)) = y$$

f - f trick

$$P(f(x), y) \quad f(f(f(x)^2 + f(y))) = f(f(x) \cdot f(x) + y)$$

$$\Rightarrow \underline{f(x^2 + f(y)) = f(x) \cdot x + y}$$

$$\cancel{f(f(x)^2 + f(y))} = \cancel{f(x^2 + f(y))}$$

$$\cancel{f(x)^2 + \cancel{f(y)}} = x^2 + \cancel{f(y)}$$

$$f(a) = a \quad f(b) = -b \quad a \neq 0, b \neq 0$$

$$3) \quad X_{n+1} = 6X_n + 2 \sum_{i=0}^n X_i \quad X_0 = 1$$

$$X_n = 6X_{n-1} + 2 \sum_{i=0}^{n-1} X_i \quad X_1 = 8$$

$$\leadsto 2 \sum_{i=0}^{n-1} X_i = X_n - 6X_{n-1}$$

$$\Rightarrow X_{n+1} = 6X_n + 2X_n + X_n - 6X_{n-1}$$

$$\Rightarrow \begin{cases} X_{n+1} = 9X_n - 6X_{n-1} \\ X_0 = 1 \quad X_1 = 8 \end{cases}$$

$$\lambda^2 - 9\lambda + 6 = 0$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{57}}{2}$$

$$\leadsto X_n = A \lambda_1^n + B \lambda_2^n$$

$$A+B=1$$

$$A\lambda_1 + B\lambda_2 = 8$$

→ pensare alla condizione

$$1) f: \mathbb{Q} \rightarrow \mathbb{Q} \quad \underline{f(x + f(y)) = f(x) + y}$$

$$P(x, y) \quad f(f(y)) = \overset{0}{f(0)} + y$$

$$f(x) = -x$$

f è suriettiva e f è iniettiva

$$\exists t \mid f(t) = 0 \quad f(x) = f(x) + t \Rightarrow t = 0$$

$$y = f(y) \quad f(x+y) = f(x) + f(y)$$

$$2) f(2a) + 2f(b) = f(f(a+b)) \quad f: \mathbb{Q} \rightarrow \mathbb{Q}$$

$$-2f(a) - 2f(b)$$

ho simmetria a Sx e a Sx si

$$f(2b) + 2f(a) = f(f(a+b)) = f(2a) + 2f(b)$$

$$\Rightarrow 2f(a) - f(2a) = 2f(b) - f(2b)$$

$$2f(x) - f(2x) = \text{costante} = k$$

$$f(b) = k \quad -k = f(f(a+b)) - 2f(b) - 2f(a)$$

$$b \Rightarrow \boxed{f(2a) + 2k = f(f(a))}$$

$$f(2(a+b)) + 2k = f(f(a+b))$$

$$k = f(2(a+b)) + 2k - f(2a) - f(2b)$$

$$f(a'+b') = f(a') + f(b') - k$$

$$g(x) = f(x) - k$$

$$\underbrace{f(a'+b') - k}_{g(a'+b')} = \underbrace{f(a') - k}_{g(a')} + \underbrace{f(b') - k}_{g(b')}$$

$$g(a' + b') = g(a') + g(b') \quad g(x) = c \cdot x$$

$$f(x) = c \cdot x + k \leadsto \text{poi verificati se funziunano}$$

$$f(x) = 0 \vee f(x) = 2x + k$$

$$f(f(n)) = n + 2025 \quad f: \mathbb{Z} \rightarrow \mathbb{Z}$$